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SPECIAL FINITE ELEMENT METHODS FOR A CLASS OF SECOND ORDER ELLIPTIC PROBLEMS WITH ROUGH COEFFICIENTS

by

Ivo Babuška

Gabriel Caloz

and

John E. Osborn

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SPECIAL FINITE ELEMENT METHODS FOR A CLASS OF SECOND ORDER ELLIPTIC PROBLEMS WITH ROUGH COEFFICIENTS

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Ivo Babuška* Gabriel Caloz† John E. Osborn#

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*Institute for Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, MD 20742. The work of this author was supported by the U.S. Office of Naval Research under Contract N00014-90-J-1030 and by the National Science Foundation under Grant CCR-88-20279.

†Institut de Recherche Mathématique de Rennes, Université de Rennes I, 35042 Rennes Cedex, France. The work of this author was partially supported by the Swiss National Science Foundation.

#Department of Mathematics, University of Maryland, College Park, MD 20742. The work of this author was partially supported by the National Science Foundation under Grant # DMS-84-10324 and by the Institute for Physical Science and Technology, University of Maryland, College Park.

Abstract

In this paper we consider the approximate solution of a class of second order elliptic equations with rough coefficients. Problems of the type considered arise in the analysis of unidirectional composites, where the coefficients represent the properties of the material. We present several methods for this class of problems, and show that they have the same accuracy as usual methods have for problems with smooth coefficients. We refer to the methods as special finite elements methods because they are of finite element type but employ special shape functions, chosen to effectively model the unknown solution.

Key words: special finite elements, rough coefficients

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Special Finite Element Methods for a Class of

Second Order Elliptic Problems with Rough Coefficients

1. Introduction

In this paper we consider the approximate solution of a class of second order, two dimensional elliptic boundary value problems with rough or highly oscillating coefficients. We apply an approach proposed by Babuška and Osborn [5] for the approximate solution of problems with rough input data. This approach was applied to one dimensional problems in Babuška and Osborn [4]. Specifically, we consider boundary value problems of the form

(1.1)

$$\begin{cases} Lu(x,y) \equiv -\frac{\partial}{\partial x}(a(x,y)\frac{\partial}{\partial x}u(x,y)) - \frac{\partial}{\partial y}(a(x,y)\frac{\partial}{\partial y}u(x,y)) = f(x,y) \quad \forall\ (x,y) \in \Omega \\ u(x,y) = 0 \quad \forall\ (x,y) \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, f is a function in $L^2(\Omega)$, and where the function $a \in L^\infty(\Omega)$ satisfies

$$0 < \alpha \le a(x,y) \le \beta < \infty \quad \forall\ (x,y) \in \Omega, \tag{1.2}$$

where α and β are constants. Throughout most of the paper we will also assume that $a(x,y)$ locally varies sharply in at most one direction, a requirement on the coefficient a that will be made precise later (see Remarks 2.1 and 4.1); such coefficients will also be called (curvilinear or straight line) unidirectional. If the coefficient a is rough, then the solution u to (1.1) will also be rough; to be specific, u will not in general be in $H^2(\Omega)$ and may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Problems of this type arise in many applications; we will be especially concerned with applications to unidirectional composite materials (briefly, composites). In these applications the coefficient $a(x,y)$ represents the

properties of the material, and changes abruptly. We will also be interested in problems in which $a(x,y)$ changes smoothly but rapidly. We will take the liberty of referring to both types of problems as composites. In Figs. 1.1-1.4 we show some typical configurations for unidirectional composites. In these figures the coefficient is constant or is changing slowly along the lines or curves and is changing sharply in the transverse direction; the

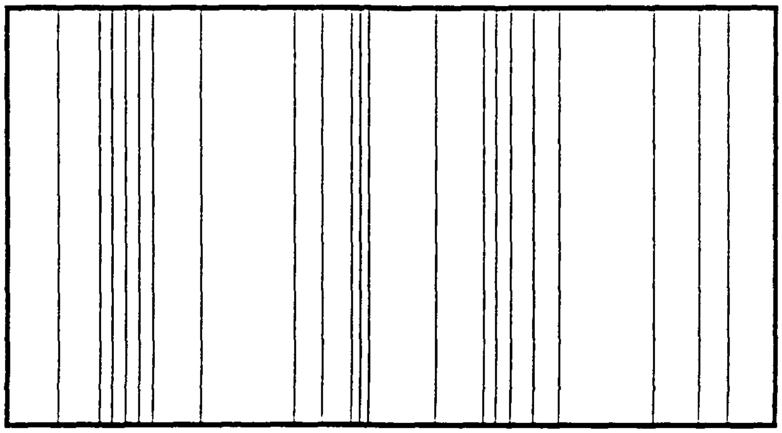

Fig. 1.1. A Straight Line Unidirectional Composite

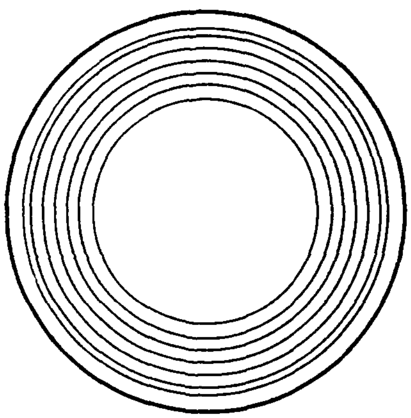

Fig. 1.2. A Tubular (Curvilinear Unidirectional) Composite

absence of lines in a portion of the material indicates a constant or a slowly varying coefficient. We can interpret the lines as fibers in the composite. This interpretation is, of course, symbolic for problems in which $a(x,y)$ changes smoothly but rapidly.

Fig. 1.1 shows a straight line unidirectional composite or coefficient and Fig. 1.2 shows the cross-section of a tubular composite. Figs. 1.3a,b show reinforced panels. The area A in Fig. 1.3b indicates a region in which

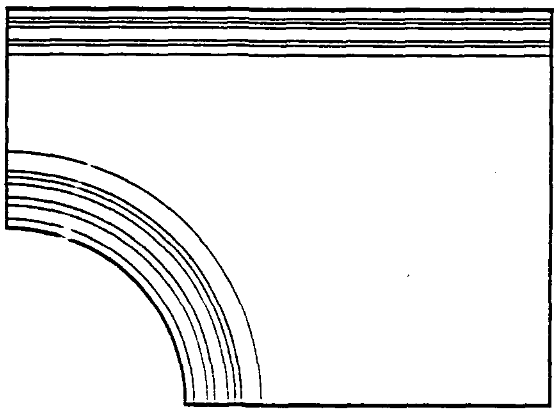

Fig. 1.3a. Reinforced Panel

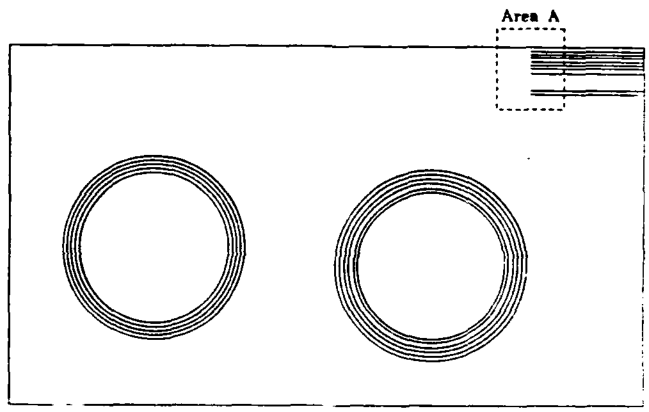

Fig. 1.3b. Reinforced Panel

$a(x,y)$ is smoothly becoming a constant or a smooth function. We refer to the materials or the coefficients in Figs. 1.2-1.3 as curvilinear unidirectional. Still more complicated problems can be considered. An example is shown in Fig. 1.4. While Figs. 1.1-1.3 depict problems in which $a(x,y)$ is unidirectional everywhere in the domain, the problem depicted in Fig. 1.4 corresponds to a coefficient that is unidirectional except on certain lines. Hence we

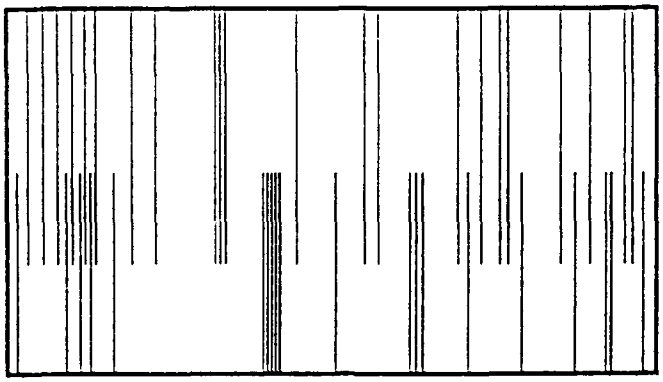

Fig. 1.4 Irregular Unidirectional Composite

call it an irregular unidirectional composite. This type of problem will be addressed in detail in a forthcoming paper. We note that certain interface problems can be naturally treated as problems of composites. With this approach it is not necessary to fit the interface with the finite elements, as is done with the standard approach.

A finite element method is obtained by restricting the weak formulation of problem (1.1),

$$(1.3)\qquad \begin{cases} u \in H_0^1(\Omega) \\ B(u,v) \equiv \int_\Omega a \, \underset{\sim}{\text{grad}}\, u \cdot \underset{\sim}{\text{grad}}\, v \, dxdy = \int_\Omega fv \, dxdy \quad \forall\, v \in H_0^1(\Omega), \end{cases}$$

to finite dimensional trial and test spaces. The outline of the approach given in [5] is as follows:

1) Characterize the space of solutions corresponding to the space of right-hand sides (in our case we suppose $f \in L^2(\Omega)$). This will involve a regularity result. Although regularity results are well-known for elliptic problems with smooth coefficients, they are not available in a direct form for our problem. Such results will be discussed in Section 2.

2) Select trial spaces that have good approximation properties. The approximation properties of the trial functions or shape functions are directly tied to the regularity of the solution. For example, if the solution u of (1.1) is not in $H^2(\Omega)$, then it is well-known that the usual finite element method based on piecewise linear approximating functions produces inaccurate results. The problem of selecting optimal trial functions is not simple; in practice, one would like to find a trial space that performs almost as well as the optimal one but that can be reasonably implemented. We use the phrase special finite element methods to denote methods with this type of special shape functions.

3) Select a test space so as to ensure the inf-sup (or stability) condition is satisfied and so that the resulting finite element method can be reasonably implemented.

We will use this approach to design methods of finite element type which will yield, roughly speaking, the same accuracy as the usual finite element method when a is smooth, but strikingly improved accuracy when a is rough.

The organization of the paper is as follows. In Section 2 we present the regularity results needed for the problems we are dealing with. (Although Theorem 2.4 presents a regularity result for problems of the type depicted in Fig. 1.4, the complete treatment of which will, as mentioned above, be the subject of a forthcoming paper, we have included it for completeness.) Then we will propose and analyze several methods to solve problem (1.1) in the special case in which $\Omega = \Omega_0 = (0,1) \times (0,1)$ and $a(x,y) = a(x)$ is a function of x only. This study is carried out in Section 3, where we propose three distinct approximation methods. A function $a(x,y) = a(x)$ of x only is an example of a function that locally varies sharply in at most one direction; in fact, such an $a(x,y)$ globally varies sharply in at most one

direction. $a(x,y)$ can also be referred to as straight line unidirectional (see Fig. 1.1). In Section 4 we present a further development of two of the methods from Section 3 in order to treat problems of the type depicted in Figs. 1.2 and 1.3 with curvilinear unidirectional coefficients.

As noted above the approach presented in this paper is thoroughly studied in the one-dimensional case in [4]. Techniques similar to special elements were used in Ciarlet, Natterer, and Varga [8] and in Crouzeix and Thomas [9] to handle degenerate one-dimensional elliptic problems. We also mention the recent work of Moussaoui and Ziani [16], which deals with the same kind of problems with a method similar to our Method I, presented in Subsection 3.1. Finally we mention the papers [3, 17, 18], which are related to our approach.

Throughout the paper, we will use the $L^2(\Omega)$-based Sobolev spaces $H^k(\Omega)$, consisting of functions with partial derivatives of order less than or equal to k in $L^2(\Omega)$. These spaces are equipped with the norms and semi-norms

$$\|u\|_{k,\Omega}^2 = \int_\Omega \sum_{|\alpha|\le k} |D^\alpha u|^2,$$

$$|u|_{k,\Omega}^2 = \int_\Omega \sum_{|\alpha|=k} |D^\alpha u|^2.$$

We will also use the spaces $H^k(\Omega)$ for fractional k. $H_0^1(\Omega)$ consists of those functions in $H^1(\Omega)$ which vanish on $\partial\Omega$. We will also use the space $H^{-1}(\Omega) = [H_0^1(\Omega)]'$. Throughout, C will denote a generic constant. When we say "there exists $C = C(\alpha,\beta)$," we mean that C depends on the coefficient $a(x,y)$ only through its upper and lower bounds α and β (cf. (1.2)).

The authors would like to thank Professor L. C. Evans for calling the theorem of S. N. Berstein used in Section 2 to their attention.

2. Regularity Results

It is clear that problem (1.1) has a unique (weak) solution in $H_0^1(\Omega)$; cf. (1.3). This is an immediate consequence of the Lax-Milgram theorem. Furthermore,

$$\|u\|_{1,\Omega} \le C(\alpha)\|f\|_{0,\Omega}.$$

But if $a(x,y)$ is rough, then u may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$, and therefore we cannot expect any reasonable rate of convergence for the usual finite element method. Nevertheless, as a consequence of the assumption that $a(x,y)$ is unidirectional, the solution u does satisfy a regularity property that can be employed in the derivation of an approximation method for (1.1) with a good rate of convergence, even though $a(x,y)$ is rough.

It is the purpose of this section to prove such regularity results, first for the model problem consisting of (1.1) with $\Omega = \Omega_0 = (0,1) \times (0,1)$ and the coefficient $a(x,y)$ satisfying $a(x,y) = a(x)$, i.e., with a straight line unidirectional coefficient (cf. Fig. 1.1), and then for the more general problem with a curvilinear unidirectional coefficient (cf. Figs 1.2-1.3). Finally we prove a regularity result that will be applied in a forthcoming paper to a problem with an irregular unidirectional coefficient (cf. Fig. 1.4). Our main tool is a theorem of Bernstein [6], [13, Section 3.17] for elliptic equations in non-divergence form, which we now state.

Consider the problem

$$\text{(2.1)} \qquad \begin{cases} -a_{11}\dfrac{\partial^2 u}{\partial x^2} - 2a_{12}\dfrac{\partial^2 u}{\partial x \partial y} - a_{22}\dfrac{\partial^2 u}{\partial y^2} = f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with a Lipschitz and piecewise C^2 boundary $\partial\Omega$ and where the functions $a_{ij} \in L^\infty(\Omega)$ satisfy

$$(2.2)\qquad \nu \sum_{i=1}^{2} \xi_i^2 \le \sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j \le \mu \sum_{i=1}^{2} \xi_i^2 \quad \forall\ (x,y) \in \Omega, \quad \forall\ \underset{\sim}{\xi} \in \mathbb{R}^2,$$

with $a_{21} = a_{12}$, where ν and μ are positive constants. Note that the equation in (2.1) is in non-divergence form.

Theorem 2.1 (Bernstein). For each $f \in L^2(\Omega)$, problem (2.1) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, there is a constant $C(\nu,\mu)$, depending on ν and μ but independent of f, such that

$$(2.3)\qquad \|u\|_{2,\Omega} \le C(\nu,\mu)\|f\|_{0,\Omega}. \qquad \blacksquare$$

Our hypothesis on Ω is not identical to the one in [13]. To prove that (2.3) is still valid for such a domain, one can use the a priori estimates given in [11, Section 3.1].

The first application of Bernstein's Theorem will give a regularity result for problem (1.1) when $\Omega = \Omega_0$ and $a(x,y) = a(x)$. Corresponding to problem (1.1), with this assumption, we define the space

$$(2.4)\qquad H^L(\Omega) = \{u \in H^1(\Omega) : a(x)\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y} \in H^1(\Omega)\}$$

with the norm

$$(2.5a)\qquad \|u\|_{L,\Omega}^2 = \|u\|_{1,\Omega}^2 + |u|_{L,\Omega}^2,$$

where

$$(2.5b)\qquad |u|_{L,\Omega}^2 = \int_\Omega \left[a\left|\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)\right|^2 + a\left|\frac{\partial^2 u}{\partial x\partial y}\right|^2 + \frac{1}{a}\left|\frac{\partial^2 u}{\partial y^2}\right|^2\right]dxdy.$$

Theorem 2.2. Suppose $\Omega = \Omega_0$ and $a(x,y) = a(x)$. Then for each $f \in L^2(\Omega)$ the solution u of (1.1) is in $H_0^1(\Omega) \cap H^L(\Omega)$. Furthermore, there is a constant $C = C(\alpha,\beta)$, depending on α and β but independent of f, such

that

(2.6) $$\|u\|_{L,\Omega} \le C(\alpha,\beta)\|f\|_{0,\Omega}.$$

Proof. Let u be the unique solution to (1.1) in $H_0^1(\Omega)$. We introduce the change of variables or mapping

(2.7) $$\tilde{x}(x) = \int_0^x \frac{ds}{a(s)}, \quad \tilde{y}(y) = y$$

and the notation

(2.8) $$\tilde{u}(\tilde{x}(x),\tilde{y}(y)) = u(x,y), \quad (x,y) \in \Omega.$$

The mapping (2.7) maps the domain Ω onto $\tilde{\Omega} = (0,\int_0^1 \frac{ds}{a(s)})\times(0,1)$. We see that $\tilde{u} \in H^2(\tilde{\Omega})$ if and only if $u \in H^1(\Omega)$, $a\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y} \in H^1(\Omega)$, which is equivalent to $u \in H^L(\Omega)$. We also note that the weak formulation (1.3) of (1.1) is transformed into

(2.9) $$\begin{cases} \tilde{u} \in H_0^1(\tilde{\Omega}) \\ \int_{\tilde{\Omega}} \left(\frac{\partial\tilde{u}}{\partial\tilde{x}}\frac{\partial\tilde{v}}{\partial\tilde{x}} + \tilde{a}^2 \frac{\partial\tilde{u}}{\partial\tilde{y}}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right) d\tilde{x}d\tilde{y} = \int_{\tilde{\Omega}} \tilde{f}\tilde{a}\tilde{v}d\tilde{x}d\tilde{y} \quad \forall\ \tilde{v} \in H_0^1(\tilde{\Omega}). \end{cases}$$

The system (2.9) is simply the variational formulation of

(2.10) $$\begin{cases} -\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} - \tilde{a}^2\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2} = \tilde{a}\tilde{f} \quad \text{in } \tilde{\Omega} \\ \tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega}. \end{cases}$$

Note that while the equation in (1.1) is in divergence form, the equation in (2.10) is in non-divergence (as well as in divergence) form. As a consequence of Theorem 2.1, (2.10) is uniquely solvable in $H_0^1(\tilde{\Omega}) \cap H^2(\tilde{\Omega})$ and

(2.11) $$\|\tilde{u}\|_{2,\tilde{\Omega}} \le C(\alpha,\beta)\|\tilde{a}\tilde{f}\|_{0,\tilde{\Omega}}.$$

Since (2.10) is uniquely solvable in $H_0^1(\tilde{\Omega})$, we conclude that $\tilde{u}$, as defined

in (2.8), which satisfies (2.9), coincides with the solution of (2.10) and hence lies in $H_0^1(\tilde{\Omega}) \cap H^2(\tilde{\Omega})$ and satisfies (2.11). Thus $u \in H_0^1(\Omega) \cap H^L(\Omega)$, which is the first conclusion in the theorem. If we change variables in the estimate (2.11) to return to the original variables, we obtain

$$\begin{aligned}
\|u\|_{L,\Omega}^2 &= \int_\Omega u^2 dxdy + \int_\Omega \left(|\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2\right) dxdy \\
&\quad + \int_\Omega \left(a|\frac{\partial}{\partial x}(a\frac{\partial u}{\partial x})|^2 + a|\frac{\partial^2 u}{\partial x \partial y}|^2 + \frac{1}{a}|\frac{\partial^2 u}{\partial y^2}|^2\right) dxdy \\
&= \int_{\tilde{\Omega}} \tilde{u}^2 \tilde{a} d\tilde{x} d\tilde{y} + \int_{\tilde{\Omega}} \left(\frac{1}{\tilde{a}} |\frac{\partial \tilde{u}}{\partial \tilde{x}}|^2 + \tilde{a}|\frac{\partial \tilde{u}}{\partial \tilde{y}}|^2\right) d\tilde{x} d\tilde{y} \\
&\quad + \int_{\tilde{\Omega}} \left(|\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}|^2 + |\frac{\partial^2 \tilde{u}}{\partial \tilde{x} \partial \tilde{y}}|^2 + |\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}|^2\right) d\tilde{x} d\tilde{y} \\
&\le \max(\beta, \frac{1}{\alpha}) \|\tilde{u}\|_{2,\tilde{\Omega}}^2 \\
&\le \max(\beta, \frac{1}{\alpha}) C^2(\alpha,\beta) \|\tilde{a}\tilde{f}\|_{0,\tilde{\Omega}}^2 \\
&\le \beta^2 \max(\beta, \frac{1}{\alpha}) C^2(\alpha,\beta) \|f\|_{0,\Omega}^2,
\end{aligned}$$

which is (2.6). ∎

Theorem 2.2 was proved by making a global change of variables and then applying the Bernstein result. The global change of variables exists because $a(x,y)$ globally varies sharply in one direction: $a(x,y) = a(x)$. We now prove a second result in which we assume the existence of only a local change of variables (cf. Figs. 1.2-1.3).

Let $\Sigma \subset \Omega$ be open and assume that we have a system of orthogonal curvilinear coordinates (ξ,η) defined on Σ. More precisely, regarding Σ and (ξ,η) we assume

(i) the functions ξ,η are defined on $\bar{\Sigma}$ and are twice continuously

differentiable,

(ii) $(\xi,\eta) : \Sigma \to \Sigma'$ is one-to-one and onto,

(iii) $\dfrac{\partial(\xi,\eta)}{\partial(x,y)} \geq \gamma > 0$ on Σ,

(iv) $\underset{\sim}{\mathrm{grad}}\ \xi \cdot \underset{\sim}{\mathrm{grad}}\ \eta = 0$ in Σ,

(v) Σ is a rectangle in ξ,η, i.e., $\Sigma' = (\xi_\Sigma^1, \xi_\Sigma^2) \times (\eta_\Sigma^1, \eta_\Sigma^2)$, and

(vi) $\bar{\Sigma} \cap \partial\Omega = \begin{cases} \emptyset, \text{ in which case all edges of } \Sigma \text{ are called interior edges} \\ \text{or} \\ \text{the union of one or more edges of } \Sigma, \text{ in which case these edges are called boundary edges and the remaining edges are called interior edges.} \end{cases}$

The union of the interior edges is denoted by E. We suppose further that

(2.12) $$a(x,y) = a'(\xi) \quad \forall\ (x,y) \in \Sigma,$$

where we use the notation, for any function w defined in Σ,

$$w'(\xi(x,y),\eta(x,y)) = w(x,y), \quad (x,y) \in \Sigma.$$

See Figs. 2.1a,b for typical configurations.

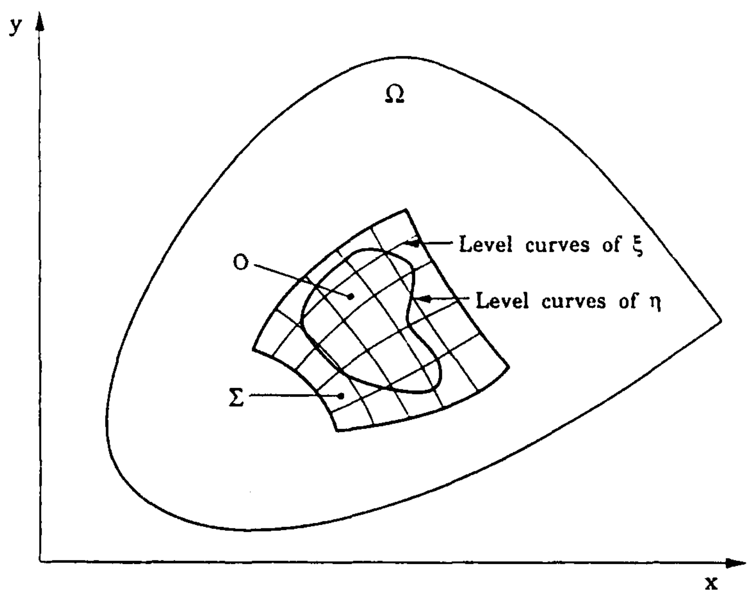

Fig. 2.1a

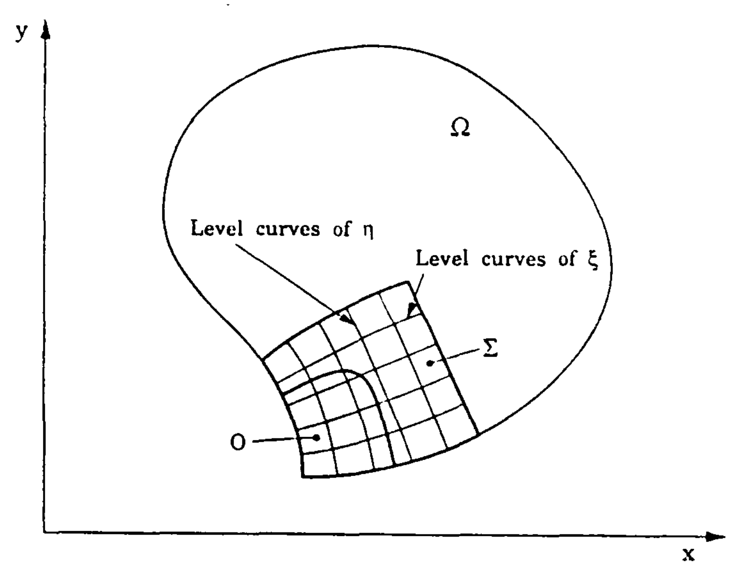

Fig. 2.1b

Theorem 2.3. Let u be the solution in $H_0^1(\Omega)$ of (1.1), where we assume that $f \in L^2(\Omega)$ and that $a(x,y)$ satisfies assumptions (1.2) and (2.12), where $\Sigma, (\xi, \eta)$ satisfies conditions (i) - (vi) above. Let $\mathcal{O} \subset \Sigma$ be open and satisfy $\mathcal{O} \subset\subset \Sigma$ if $\bar{\Sigma} \cap \partial \Omega = \emptyset$ and $\partial\mathcal{O} \cap \partial\Sigma \subset \partial\Omega$ if $\bar{\Sigma} \cap \partial\Omega \neq \emptyset$, and let $\mathcal{O}'$ be the image of $\mathcal{O}$ under the mapping (ξ, η) (cf. Figs. 2.1a,b). Then there is a constant $C = C(\alpha, \beta, \xi, \eta, d)$ depending on α, β, ξ, η, and d but independent of f, such that

$$(2.13)\quad \left[\iint_{\mathcal{O}'} \left[a'\left|\frac{\partial}{\partial\xi}\left(a'\frac{\partial u'}{\partial\xi}\right)\right|^2 + a'\left|\frac{\partial^2 u'}{\partial\xi\partial\eta}\right|^2 + \frac{1}{a'}\left|\frac{\partial^2 u'}{\partial\eta^2}\right|^2\right] d\xi d\eta\right]^{1/2} \le C(\alpha,\beta,\xi,\eta,d)\|f\|_{0,\Omega},$$

where $d = \begin{cases} \operatorname{dist}(\mathcal{O}, E), \text{ if } E \neq \emptyset \\ 1, \text{ if } E = \emptyset \end{cases}.$

Proof. Clearly u (more precisely, $u|_\Sigma$) satisfies

$$(2.14)\quad \begin{cases} u \in H^1(\Sigma) \\ \displaystyle\int_\Sigma a \operatorname{grad} u \cdot \operatorname{grad} v \, dxdy = \int_\Sigma fv \, dxdy \quad \forall\, v \in H_0^1(\Sigma). \end{cases}$$

Introducing the change of variables (ξ, η) in (2.14) we get

$$(2.15)\quad \int_{\Sigma'} a'\left(\frac{\partial u'}{\partial\xi}\frac{\partial v'}{\partial\xi}|\operatorname{grad} \xi|^2 + \frac{\partial u'}{\partial\eta}\frac{\partial v'}{\partial\eta}|\operatorname{grad} \eta|^2\right)\frac{\partial(x,y)}{\partial(\xi,\eta)}\, d\xi d\eta$$

$$= \int_{\Sigma'} f'v'\frac{\partial(x,y)}{\partial(\xi,\eta)}\, d\xi d\eta \;\forall\, v' \in H_0^1(\Sigma').$$

Now we introduce a second change of variables,

$$(2.16)\quad \tilde{\xi} = \int_{\xi_1^\Sigma}^{\xi} \frac{dt}{a'(t)}, \quad \tilde{\eta} = \eta,$$

where ξ_1^Σ is the ξ-coordinate of the left edge of Σ'. We will use the notation

$\tilde{w}'(\tilde{\xi}(\xi,\eta),\ \tilde{\eta}(\xi,\eta)) = w'(\xi,\eta),\ (\xi,\eta) \in \Sigma'$, for any function w' on Σ'

and

$$\tilde{\Sigma}' = \text{image of } \Sigma' \text{ under the mapping } (\tilde{\xi},\tilde{\eta}).$$

Applying this change of variables to (2.15) we get

$$(2.17)\quad \int_{\tilde{\Sigma}'} \left[\tilde{a}_1' \frac{\partial \tilde{u}'}{\partial \tilde{\xi}} \frac{\partial \tilde{v}'}{\partial \tilde{\xi}} + \tilde{a}'^2 \tilde{a}_2' \frac{\partial \tilde{u}'}{\partial \tilde{\eta}} \frac{\partial \tilde{v}'}{\partial \tilde{\eta}}\right] d\tilde{\xi} d\tilde{\eta} = \int_{\tilde{\Sigma}'} \tilde{f}' \tilde{v}' \tilde{a}' \frac{\partial(\tilde{x,y})}{\partial(\xi,\eta)}\, d\tilde{\xi} d\tilde{\eta}\ \forall\ \tilde{v}' \in H_0^1(\tilde{\Sigma}'),$$

where

$$(2.18)\qquad a_1'(\xi,\eta) = |\underset{\sim}{\text{grad}}\ \xi|^2 \frac{\partial(x,y)}{\partial(\xi,\eta)},\quad a_2'(\xi,\eta) = |\underset{\sim}{\text{grad}}\ \eta|^2 \frac{\partial(x,y)}{\partial(\xi,\eta)}.$$

To apply Theorem 2.1, we need to introduce Dirichlet boundary conditions. From condition (vi) we know that any edge of Σ is either an interior edge or a boundary edge. Then, through the correspondence determined by the mappings (ξ,η) and $(\tilde{\xi},\ \tilde{\eta})$, we will refer to the interior and boundary edges of Σ' and $\tilde{\Sigma}'$. Now let $\tilde{\phi}' \in C^\infty(\overline{\tilde{\Sigma}'})$ with $\tilde{\phi}'(\tilde{\xi},\tilde{\eta}) = 0$ for $(\tilde{\xi},\tilde{\eta})$ near the interior edges of $\tilde{\Sigma}'$. Then for $\tilde{v}' \in H_0^1(\tilde{\Sigma}')$, $\tilde{\phi}'\tilde{v}' \in H_0^1(\tilde{\Sigma}')$ and we can replace $\tilde{v}'$ by $\tilde{\phi}'\tilde{v}'$ in (2.17) to get

$$\int_{\tilde{\Sigma}} \left[\tilde{a}_1' \frac{\partial \tilde{u}'}{\partial \tilde{\xi}} \left[\tilde{\phi}' \frac{\partial \tilde{v}'}{\partial \tilde{\xi}} + \tilde{v}' \frac{\partial \tilde{\phi}'}{\partial \tilde{\xi}}\right] + \tilde{a}'^2 \tilde{a}_2 \frac{\partial \tilde{u}'}{\partial \tilde{\eta}} \left[\tilde{\phi}' \frac{\partial \tilde{v}'}{\partial \tilde{\eta}} + \tilde{v}' \frac{\partial \tilde{\phi}'}{\partial \tilde{\eta}}\right]\right] d\tilde{\xi} d\tilde{\eta}$$

$$= \int_{\tilde{\Sigma}'} \tilde{f}' \tilde{v}' \tilde{\phi}' \tilde{a}'\ \frac{\partial(\tilde{x,y})}{\partial(\xi,\eta)}\ d\tilde{\xi} d\tilde{\eta}$$

or

$$(2.19)\qquad \int_{\tilde{\Sigma}'} \left[\tilde{a}_1' \frac{\partial}{\partial \tilde{\xi}} (\tilde{\phi}'\tilde{u}') \frac{\partial \tilde{v}'}{\partial \tilde{\xi}} + \tilde{a}'^2 \tilde{a}_2' \frac{\partial}{\partial \tilde{\eta}} (\tilde{\phi}'\tilde{u}') \frac{\partial \tilde{v}'}{\partial \tilde{\eta}}\right] d\tilde{\xi} d\tilde{\eta} = \int_{\tilde{\Sigma}'} F\tilde{v}'\ d\tilde{\xi} d\tilde{\eta},$$

where

(2.20) $$F = \tilde{f}'\tilde{\phi}'\tilde{a}' \frac{\partial(\tilde{x},y)}{\partial(\xi,\eta)} - 2\tilde{a}_1' \frac{\partial\tilde{u}'}{\partial\tilde{\xi}} \frac{\partial\tilde{\phi}'}{\partial\tilde{\xi}} - 2\tilde{a}'^2\tilde{a}_2' \frac{\partial\tilde{u}'}{\partial\tilde{\eta}} \frac{\partial\tilde{\phi}'}{\partial\tilde{\eta}}$$

$$- \tilde{u}' \operatorname{div}_{\tilde{\xi},\tilde{\eta}}\left[\tilde{a}_1' \frac{\partial\tilde{\phi}'}{\partial\tilde{\xi}},\ \tilde{a}'^2\tilde{a}_2' \frac{\partial\tilde{\phi}'}{\partial\tilde{\eta}}\right] .$$

Writing $w = \tilde{u}'\tilde{\phi}'$, from (2.19) we get

(2.21) $$\begin{cases} w \in H_0^1(\tilde{\Sigma}') \\ \int_{\tilde{\Sigma}'} \left[\tilde{a}_1' \frac{\partial w}{\partial\tilde{\xi}} \frac{\partial\tilde{v}'}{\partial\tilde{\xi}} + \tilde{a}'^2\tilde{a}_2' \frac{\partial w}{\partial\tilde{\eta}} \frac{\partial\tilde{v}'}{\partial\tilde{\eta}}\right] d\tilde{\xi}d\tilde{\eta} = \int_{\tilde{\Sigma}'} F\tilde{v}'\, d\tilde{\xi}d\tilde{\eta} \quad \forall\ \tilde{v}' \in H_0^1(\tilde{\Sigma}'). \end{cases}$$

$w \in H_0^1(\tilde{\Sigma}')$ since $\tilde{\phi}' = 0$ near the interior edges of $\tilde{\Sigma}'$. Since $\tilde{u}'$ is in $H^1(\tilde{\Sigma}')$, the functions ξ, η are C^2, and $\tilde{a}' = \tilde{a}'(\tilde{\xi})$, we see that F is in $L^2(\tilde{\Sigma}')$. The system (2.21) is simply the variational formulation of

(2.22) $$\begin{cases} -\frac{\partial}{\partial\tilde{\xi}}\left[\tilde{a}_1' \frac{\partial w}{\partial\tilde{\xi}}\right] - \tilde{a}'^2(\tilde{\xi})\frac{\partial}{\partial\tilde{\eta}}\left[\tilde{a}_2' \frac{\partial w}{\partial\tilde{\eta}}\right] = F \quad \text{in} \quad \tilde{\Sigma}' \\ w = 0 \quad \text{on} \quad \partial\tilde{\Sigma}' . \end{cases}$$

The equation in (2.22) can be formally written as

(2.23) $$-\tilde{a}_1' \frac{\partial^2 w}{\partial\tilde{\xi}^2} - \tilde{a}'^2\tilde{a}_2' \frac{\partial^2 w}{\partial\tilde{\eta}^2} = F + \frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}} \frac{\partial w}{\partial\tilde{\xi}} + \tilde{a}'^2 \frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}} \frac{\partial w}{\partial\tilde{\eta}} \equiv G.$$

Denote by W the unique solution in $H^2(\tilde{\Sigma}') \cap H_0^1(\tilde{\Sigma}')$ of

(2.24) $$-\tilde{a}_1' \frac{\partial^2 W}{\partial\tilde{\xi}^2} - \tilde{a}'^2\tilde{a}_2' \frac{\partial^2 W}{\partial\tilde{\eta}^2} = G,$$

which exists by Theorem 2.1 and which satisfies

(2.25) $$\|W\|_{2,\tilde{\Sigma}'} \leq C(\alpha,\beta,\xi,\eta)\ \|G\|_{0,\tilde{\Sigma}'}.$$

Now $w \in H_0^1(\tilde{\Sigma}')$ solves the same problem formally. We want to show that $w = W$, and hence that $w \in H^2(\tilde{\Sigma}')$ and satisfies (2.25). Writing (2.24) in divergence form we obtain

$$-\frac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\frac{\partial W}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\frac{\partial W}{\partial\tilde{\eta}}\right)$$

$$(2.26) \qquad = -\tilde{a}_1'\frac{\partial^2 W}{\partial\tilde{\xi}^2} - \tilde{a}'^2\tilde{a}_2'\frac{\partial^2 W}{\partial\tilde{\eta}^2} - \frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial W}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial W}{\partial\tilde{\eta}}$$

$$= F + \frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial w}{\partial\tilde{\xi}} + \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial w}{\partial\tilde{\eta}} - \frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial W}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial W}{\partial\tilde{\eta}}\,.$$

Letting $U = W - w$ and using (2.22) and (2.26) we see that

$$(2.27) \qquad \begin{cases} -\frac{\partial}{\partial\tilde{\xi}}\left(\tilde{a}_1'\frac{\partial U}{\partial\tilde{\xi}}\right) - \tilde{a}'^2\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{a}_2'\frac{\partial U}{\partial\tilde{\eta}}\right) \\ \qquad = -\frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial U}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial U}{\partial\tilde{\eta}} \quad \text{in } \tilde{\Sigma}' \\ U = 0 \quad \text{on } \partial\tilde{\Sigma}', \end{cases}$$

where we understand the equation in the weak sense. It will suffice to show that $U = 0$.

Let $T : H^{-1}(\tilde{\Sigma}') \longrightarrow H_0^1(\tilde{\Sigma}')$ be the solution operator corresponding to the problem (2.22), i.e., let $TF = w$. Then from (2.27) we get

$$(2.28) \qquad U = T\left(-\frac{\partial\tilde{a}_1'}{\partial\tilde{\xi}}\frac{\partial U}{\partial\tilde{\xi}} - \tilde{a}'^2\frac{\partial\tilde{a}_2'}{\partial\tilde{\eta}}\frac{\partial U}{\partial\tilde{\eta}}\right) \equiv AU.$$

Since $T : H^{-1}(\tilde{\Sigma}') \longrightarrow H_0^1(\tilde{\Sigma}')$ is bounded and $H^0(\tilde{\Sigma}')$ is compactly contained in $H^{-1}(\tilde{\Sigma}')$, we see that $A : H_0^1(\tilde{\Sigma}') \longrightarrow H_0^1(\tilde{\Sigma}')$ is compact. Suppose now that $U \neq 0$. Then from (2.28) we see that 1 is an eigenvalue of A. Hence 1 is an eigenvalue of A^*; let V be an associated eigenfunction.

We can choose $V' \in H^2(\tilde{\Sigma}') \cap H_0^1(\tilde{\Sigma}')$ so that $\|V - V'\|_{1,\tilde{\Sigma}'} < \|V\|_{1,\tilde{\Sigma}'}$. Then $(V,V')_{H^1(\tilde{\Sigma}')} \neq 0$ and from the Fredholm alternative we see that the problem

(2.29)
$$(I - A)Z = V'$$

has no solution in $H_0^1(\tilde{\Sigma}')$. Recalling the definition of A we see that equation (2.29) can be written

(2.30)
$$Z - T\left[-\frac{\partial \tilde{a}_1'}{\partial \tilde{\xi}}\frac{\partial Z}{\partial \tilde{\xi}} - \tilde{a}'^{,2}\frac{\partial \tilde{a}_2'}{\partial \tilde{\eta}}\frac{\partial Z}{\partial \tilde{\eta}}\right] = V',$$

which can be formally written as

$$-\frac{\partial}{\partial \tilde{\xi}}\left(\tilde{a}_1'\frac{\partial Z}{\partial \tilde{\xi}}\right) - \tilde{a}'^{,2}\frac{\partial}{\partial \tilde{\eta}}\left(\tilde{a}_2'\frac{\partial Z}{\partial \tilde{\eta}}\right)$$

$$= -\frac{\partial \tilde{a}_1'}{\partial \tilde{\xi}}\frac{\partial Z}{\partial \tilde{\xi}} - \tilde{a}'^{,2}\frac{\partial \tilde{a}_2'}{\partial \tilde{\eta}}\frac{\partial Z}{\partial \tilde{\eta}} - \frac{\partial}{\partial \tilde{\xi}}\left(\tilde{a}_1'\frac{\partial V'}{\partial \tilde{\xi}}\right) - \tilde{a}'^{,2}\frac{\partial}{\partial \tilde{\eta}}\left(\tilde{a}_2'\frac{\partial V'}{\partial \tilde{\eta}}\right)$$

or

$$-\tilde{a}_1'\frac{\partial^2 Z}{\partial \tilde{\xi}^2} - \tilde{a}'^{,2}\tilde{a}_2'\frac{\partial^2 Z}{\partial \tilde{\eta}^2} = -\frac{\partial}{\partial \tilde{\xi}}\left(\tilde{a}_1'\frac{\partial V'}{\partial \tilde{\xi}}\right) - \tilde{a}'^{,2}\frac{\partial}{\partial \tilde{\eta}}\left(\tilde{a}_2'\frac{\partial V'}{\partial \tilde{\eta}}\right).$$

But Theorem 2.1 shows that this equation has a solution Z in $H^2(\tilde{\Sigma}') \cap H_0^1(\tilde{\Sigma}')$. It is immediate that this Z solves (2.30) and hence solves (2.29), a contradiction. Therefore we conclude that $U = 0$. We have thus shown that $w = W$ and hence that $w \in H^2(\tilde{\Sigma}')$ and

(2.25')
$$\|w\|_{2,\tilde{\Sigma}'} \le C(\alpha,\beta,\xi,\eta)\,\|G\|_{0,\tilde{\Sigma}'}.$$

Now the function $\tilde{\phi}'$ can be chosen as a cut-off function satisfying $\tilde{\phi}' = 1$ on $\tilde{\mathcal{O}}'$, $|D\tilde{\phi}'| \le C(\tilde{d}')^{-1}$, and $|D^2\tilde{\phi}'| \le C(\tilde{d}')^{-2}$, where C is some positive constant, and

$$\tilde{d}' = \begin{cases} \mathrm{dist}(\tilde{\mathcal{O}}', \tilde{E}'), & \text{if } \tilde{E}' \ne \phi \\ 1, & \text{if } \tilde{E}' = \phi \end{cases}.$$

Then from the expression for G in (2.23) and from (2.20) and (2.18) we have

$$\|G\|_{0,\tilde{\Sigma}'} \le C(\alpha,\beta,\xi,\eta)(\tilde{d}')^{-2}\,\|f\|_{0,\Omega}.$$

Thus from (2.25′) we see that

$$(2.31) \qquad \|\tilde{u}'\|_{2,\tilde{\mathcal{O}}'} \le \|w\|_{2,\tilde{\Sigma}'} \le C(\alpha,\beta,\xi,\eta,\tilde{d}')\ \|f\|_{0,\Omega}.$$

Now changing from $\tilde{\xi},\tilde{\eta}$ back to ξ,η in (2.31) we obtain

$$\left[\int_{\mathcal{O}'}\left\{a'\left|\frac{\partial}{\partial\xi}\left(a'\frac{\partial u'}{\partial\xi}\right)\right|^2 + a'\left|\frac{\partial^2 u'}{\partial\xi\partial\eta}\right|^2 + \frac{1}{a'}\left|\frac{\partial^2 u'}{\partial\eta^2}\right|^2\right\}d\xi d\eta\right]^{1/2}$$

$$\le C(\alpha,\beta,\xi,\eta,\tilde{d}')\ \|f\|_{0,\Omega},$$

which yields (2.13) since $C_1(\alpha,\beta,\xi,\eta)\tilde{d}' \le d \le C_2(\alpha,\beta,\xi,\eta)\tilde{d}'$. ■

<u>Remark 2.1</u>. Equation (2.12), with (ξ,η) and Σ satisfying the conditions (i)-(vi), is basis for the precise meaning of the phrase "a locally varies sharply in at most one direction," which is fully formulated in Subsection 4.1 (see Remark 4.1).

<u>Remark 2.2</u>. If the mapping function $\xi+i\eta$ is analytic, then the above analysis is simplified since in this case the functions a_1 and a_2 in (2.18) are equal to 1.

<u>Remark 2.3</u>. We can define the local analogue of the space $H^L(\Omega)$ defined in (2.5). With $\Sigma,(\xi,\eta)$, and $\mathcal{O}$ as in Theorem 2.3,

$$H^L(\mathcal{O}) = \{u : u' \in H^1(\mathcal{O}')\ ,\ a'\frac{\partial u'}{\partial\xi},\ \frac{\partial u'}{\partial\eta} \in H^1(\mathcal{O}')\}$$

with the norm

$$\|u\|^2_{L,\mathcal{O}} = \|u\|^2_{1,\mathcal{O}} + |u|^2_{L,\mathcal{O}},$$

where

$$|u|^2_{L,\mathcal{O}} = \int_{\mathcal{O}'}\left[a'\left|\frac{\partial}{\partial\xi}(a'\frac{\partial u'}{\partial\xi})\right|^2 + a'\left|\frac{\partial^2 u'}{\partial\xi\partial\eta}\right|^2 + \frac{1}{a'}\left|\frac{\partial^2 u'}{\partial\eta^2}\right|^2\right]d\xi d\eta.$$

In terms of the semi-norm $|u|_{L,\mathcal{O}}$, (2.13) can be stated as

$$|u|_{L,\mathcal{O}} \le C(\alpha,\beta,\xi,\eta,d)\|f\|_{0,\Omega}.$$

Remark 2.4. Theorems 2.2 and 2.3 can be easily generalized to cover coefficients $a(x,y)$ of the form $a_1(x)a_2(y)$ and $a_1(\xi)a_2(\eta)$, respectively, and coefficients $a(x,y)$ that are rough in x but smooth in y and $a(\xi,\eta)$ that are rough in ξ but smooth in η, respectively.

The method used in the proof of Theorem 2.3 gives a constant C which behaves as d^{-2}, where $d = \mathrm{dist}(\mathcal{O},S)$; this type of estimate is sufficient to treat problems of the type depicted in Figs. 1.1-1.3. To analyze the type of problem depicted in Fig. 1.4 we will need a sharper estimate. Although the complete study of these problems is reserved for a forthcoming paper, we here present the relevant regularity result; we show, in fact, that the constant C behaves as d^{-1} instead of d^{-2} by using a more refined analysis. Although the idea of the analysis is quite general, we will carry out the analysis only on a simple model problem so as to focus on the essential idea.

Let $\Omega = \Omega_0 = (0,1)\times(0,1)$ be the square with the boundary $\partial\Omega$ composed of the four straight lines Γ_i, $i = 1,\dots,4$, shown in Figure 2.2.

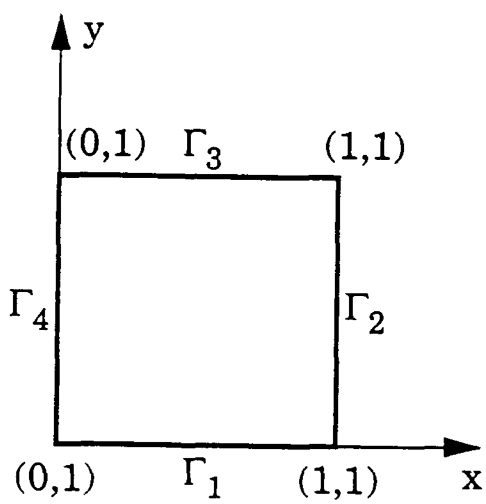

Fig. 2.2

Let us further define

$$\overset{(0)}{H}{}^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\}$$

and, for $0 < d < \frac{1}{2}$,

$$\Omega_d = \{(x,y) \in \Omega : d < y < 1\}.$$

We assume that the coefficient $a \in L^\infty(\Omega)$ is a function of x only and satisfies

$$(2.32) \qquad 0 < \alpha \le a(x) \le \beta < \infty \quad \forall\, x \in (0,1).$$

Theorem 2.4. For $f \in L^2(\Omega)$ and $a \in L^\infty(\Omega)$ satisfying (2.32), let $u \in \overset{(0)}{H}{}^1(\Omega)$ satisfy

$$(2.33) \qquad \int_\Omega a(x)\,\underset{\sim}{\mathrm{grad}}\, u\ \underset{\sim}{\mathrm{grad}}\, v\ dxdy = \int_\Omega fv\ dxdy \quad \forall\, v \in H_0^1(\Omega).$$

Then there exists $C = C(\alpha,\beta)$, depending on α and β but independent of d, such that

$$(2.34) \qquad \|u\|_{L,\Omega_d} \le C(\|f\|_{0,\Omega} + d^{-1}\|u\|_{1,\Omega}).$$

Proof. As in the proof of Theorem 2.2 we introduce the change of variables

$$(2.35) \qquad \tilde{x}(x) = \int_0^x \frac{ds}{a(s)}, \quad \tilde{y}(y) = y$$

and the notation

$$(2.36) \qquad \tilde{w}(\tilde{x}(x), \tilde{y}(y)) = w(x,y), \quad (x,y) \in \Omega.$$

For any function w in Ω, $\tilde{w}$ is defined in $\tilde{\Omega} = (0,A)\times(0,1)$ with $A = \int_0^1 \frac{ds}{a(s)}$. A tilde means that we are in the domain $\tilde{\Omega}$.

Obviously for a function u satisfying our hypotheses, $\tilde{u}$, as defined by (2.35) and (2.36), is in $\overset{(0)}{H}{}^1(\tilde{\Omega})$ and

$$C_1(\alpha,\beta)\ \|u\|_{1,\Omega} \le \|\tilde{u}\|_{1,\tilde{\Omega}} \le C_2(\alpha,\beta)\|u\|_{1,\Omega}. \tag{2.37}$$

Furthermore $\tilde{u}$ satisfies the differential equation

$$-\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} - \tilde{a}^2(\tilde{x})\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2} = \tilde{f}\tilde{a}(\tilde{x}),\ \ (\tilde{x},\tilde{y}) \in \tilde{\Omega}. \tag{2.38}$$

Here and in the remainder of this section we will interpret such equations in the weak sense. Using (2.37) we see that (2.34) holds provided we prove

$$\|\tilde{u}\|_{2,\tilde{\Omega}_d} \le C(\|\tilde{f}\|_{0,\tilde{\Omega}} + d^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}), \tag{2.39}$$

where C is independent of d.

The idea of the proof, similar to that of the proof of Theorem 2.3, is to apply Bernstein's theorem to $\tilde{u}$ times a cut-off function χ chosen so that $\chi\,\tilde{u} \in H_0^1(\tilde{\Omega})$. But to get a sharper estimate, we first extract from $\tilde{u}$ a smooth function (ϕ, defined in (2.42)) having almost the values of $\tilde{u}$ on $\tilde{\Gamma}_1$. This cannot be done in a simple way, since we only know $\tilde{u} \in H^1(\tilde{\Omega})$.

The restrictions of $\tilde{u}$ to the edge $\tilde{\Gamma}_1$ is denoted by

$$w(\tilde{x}) = \tilde{u}(\tilde{x},0),\ \ \tilde{x} \in \tilde{I} = (0,A).$$

Since we know that

$$w \in H_{00}^{1/2}(\tilde{I}) \quad \text{and} \quad \|w\|_{1/2,0,0,\tilde{I}} \le C\|\tilde{u}\|_{1,\tilde{\Omega}},$$

where $\|\cdot\|_{1/2,0,0,\tilde{I}}$ denotes the norm in the space $H_{00}^{1/2}(\tilde{I})$ (cf. [14]), we can represent w in terms of its Fourier series,

$$w(\tilde{x}) = \sum_{k=1}^{\infty} a_k \sin k\,\frac{\pi\tilde{x}}{A},$$

and the coefficients a_k will have the property

(2.40) $$\sum_{k=1}^{\infty} ka_k^2 \le C\|\tilde{u}\|_{1,\tilde{\Omega}}^2.$$

We introduce the function

(2.41) $$\rho(\tilde{x},\tilde{y}) = \sum_{k=1}^{\infty} a_k \sin k\pi\tilde{x}/A \frac{\text{sh } k\pi(1-\tilde{y})/A}{\text{sh } k\pi/A},$$

which is the solution of

$$\begin{cases} \Delta\rho = 0 & \text{in } \Omega \\ \rho = \tilde{u} & \text{on } \partial\Omega . \end{cases}$$

Then we write $\rho = \phi + \psi$, where

(2.42) $$\phi(\tilde{x},\tilde{y}) = \sum_{k=1}^{[d^{-1}]} a_k \sin k\pi\tilde{x}/A \frac{\text{sh } k\pi(1-\tilde{y})/A}{\text{sh } k\pi/A}$$

and

(2.43) $$\psi(\tilde{x},\tilde{y}) = \sum_{k=[d^{-1}]+1}^{\infty} a_k \sin k\pi\tilde{x}/A \frac{\text{sh } k\pi(1-\tilde{y})/A}{\text{sh } k\pi/A},$$

and $[d^{-1}]$ is the integer part of d^{-1}.

Our method of proving (2.39) is to write $\tilde{u} = \chi_1 + \chi_2$, where $\chi_1 \in \overset{(0)}{H}{}^1(\tilde{\Omega})$ solves

(2.44) $$\begin{cases} -\dfrac{\partial^2\chi_1}{\partial\tilde{x}^2} - \tilde{a}^2(\tilde{x})\dfrac{\partial^2\chi_1}{\partial\tilde{y}^2} = \tilde{f}\tilde{a} & \text{in } \tilde{\Omega} \\ \chi_1 = \phi \text{ on } \tilde{\Gamma}_1, \end{cases}$$

and $\chi_2 \in \overset{(\circ)}{H}{}^1(\tilde{\Omega})$ solves

$$(2.45)\qquad \begin{cases} -\dfrac{\partial^2\chi_2}{\partial\tilde{x}^2} - \tilde{a}^2(\tilde{x})\dfrac{\partial^2\chi_2}{\partial\tilde{y}^2} = 0 \quad \text{in} \quad \tilde{\Omega} \\ \chi_2 = \psi \quad \text{on} \quad \tilde{\Gamma}_1, \end{cases}$$

and to bound χ_1, χ_2 in the $H^2(\tilde{\Omega}_d)$-norm. The decomposition $\tilde{u} = \chi_1 + \chi_2$ is possible since $\tilde{u}$ satisfies (2.38). First let us study χ_1, the solution of (2.44). We write $\chi_1 = \phi + z_1$, where $z_1 \in H_0^1(\tilde{\Omega})$ satisfies

$$(2.46)\qquad -\frac{\partial^2 z_1}{\partial\tilde{x}^2} - \tilde{a}^2\frac{\partial^2 z_1}{\partial\tilde{y}^2} = F_1 \equiv \frac{\partial^2\phi}{\partial\tilde{x}^2} + \tilde{a}^2\frac{\partial^2\phi}{\partial\tilde{y}^2} + \tilde{f}\tilde{a}\ .$$

Now we bound the L^2-norm of F_1. Using the explicit formula (2.42) for ϕ, the orthogonality properties of the functions $\sin k\frac{\pi}{A}\tilde{x}$, $\cos k\frac{\pi}{A}\tilde{x}$, and the formula

$$\int_0^1 \mathrm{sh}^2\,\frac{k\pi(1-\tilde{y})}{A}\,d\tilde{y} = \frac{A}{4\pi k}\,\mathrm{sh}\,\frac{2\pi k}{A} - \frac{1}{2}\ ,$$

we obtain

$$(2.47)\qquad \left\|\frac{\partial^2\phi}{\partial\tilde{x}^2}\right\|^2_{0,\tilde{\Omega}} \le C\sum_{k=1}^{[d^{-1}]} a_k^2 k^3,$$

where C depends on $A = \int_0^1 \frac{ds}{a(s)}$ but is independent of d. Combining (2.40) and (2.47) we have

$$\left\|\frac{\partial^2\phi}{\partial\tilde{x}^2}\right\|_{0,\tilde{\Omega}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

In a similar way we get

$$(2.48)\qquad |\phi|_{2,\tilde{\Omega}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

We also see that

$$(2.49)\qquad \|\phi\|_{1,\tilde{\Omega}} \le C\,\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

From (2.46) and (2.48) we get

$$(2.50) \qquad \|F_1\|_{0,\tilde{\Omega}} \le C(d^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}} + \|\tilde{f}\|_{0,\tilde{\Omega}}).$$

Using Theorem 2.1, we see from (2.46) that $z_1 \in H^2(\tilde{\Omega})$ and that

$$(2.51) \qquad \|z_1\|_{2,\tilde{\Omega}} \le C(\alpha,\beta)\,\|F_1\|_{0,\tilde{\Omega}}.$$

Using (2.48) - (2.51) we see that χ_1 is in $H^2(\tilde{\Omega})$ and satisfies

$$(2.52) \qquad \|\chi_1\|_{2,\tilde{\Omega}} \le C(d^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}} + \|\tilde{f}\|_{0,\tilde{\Omega}}).$$

Let us now consider χ_2, the solution of (2.45). Write $\chi_2 = \psi + z_2$, where $z_2 \in H_0^1(\tilde{\Omega})$ satisfies

$$(2.53) \qquad -\frac{\partial^2 z_2}{\partial \tilde{x}^2} - \tilde{a}^2 \frac{\partial^2 z_2}{\partial \tilde{y}^2} = F_2 \equiv \frac{\partial^2 \psi}{\partial \tilde{x}^2} + \tilde{a}^2 \frac{\partial^2 \psi}{\partial \tilde{y}^2},$$

where $F_2 \in H^{-1}(\Omega)$. Since χ_2 does not satisfy homogeneous boundary conditions, we cannot apply Theorem 2.1 immediately. Let $\chi \in C^\infty(0,1)$ be a cut-off function satisfying

$$\chi(\tilde{y}) = 1 \quad \text{for} \quad d < \tilde{y} < 1,$$

$$\chi(\tilde{y}) = 0 \quad \text{for} \quad 0 < \tilde{y} < \frac{d}{2},$$

and

$$|\chi^{(j)}(\tilde{y})| \le Cd^{-j}, \quad \tilde{y} \in (0,1), \quad j = 0,1,2.$$

The function $\chi_3 = \chi\chi_2 \in H_0^1(\tilde{\Omega}_{d/2})$ is the solution of

$$(2.54) \qquad -\frac{\partial^2 \chi_3}{\partial \tilde{x}^2} - \tilde{a}^2(\tilde{x})\frac{\partial^2 \chi_3}{\partial \tilde{y}^2} = F_3 + F_4 \quad \text{in} \quad \tilde{\Omega}_{d/2},$$

where (recall that $\chi_2 = \psi + z_2$)

(2.55) $$F_3 \equiv -\frac{\partial^2}{\partial\tilde{x}^2}(\chi\psi) - \tilde{a}^2(\tilde{x})\,\frac{\partial^2}{\partial\tilde{y}^2}(\chi\psi),$$

(2.56) $$F_4 = -\frac{\partial^2}{\partial\tilde{x}^2}(\chi z_2) - \tilde{a}^2(\tilde{x})\,\frac{\partial^2}{\partial\tilde{y}^2}(\chi z_2).$$

We will now bound F_3 and F_4 in the $L^2(\tilde{\Omega}_{d/2})$-norm. From the explicit formula (2.43) for ψ we see that

$$\Big\|\frac{\partial^2\psi}{\partial\tilde{x}^2}\Big\|^2_{0,\tilde{\Omega}_{d/2}} \le C \sum_{k=[d^{-1}]+1}^{\infty} a_k^2 k^3 e^{-\pi kd/A}.$$

Taking into account (2.40) and the fact that

$$k^2 e^{-k\pi d/A} \le Cd^{-2}e^{-\pi/A} \;\forall\; k = [d^{-1}] + 1, \dots ,$$

we obtain

(2.57) $$\Big\|\frac{\partial^2\psi}{\partial\tilde{x}^2}\Big\|_{0,\tilde{\Omega}_{d/2}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

Analogously we get

(2.58) $$\Big\|\frac{\partial^2\psi}{\partial\tilde{y}^2}\Big\|_{0,\tilde{\Omega}_{d/2}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}},$$

(2.59) $$\Big\|\frac{\partial\psi}{\partial\tilde{x}}\Big\|_{0,\tilde{\Omega}}\,,\;\Big\|\frac{\partial\psi}{\partial\tilde{y}}\Big\|_{0,\tilde{\Omega}} \le C\|\tilde{u}\|_{1,\tilde{\Omega}},$$

and

(2.60) $$\|\psi\|_{0,\tilde{\Omega}_{d/2}} \le Cd\|\tilde{u}\|_{1,\tilde{\Omega}}\,,$$

with C independent of d. Using (2.55), (2.57)-(2.60), and the hypotheses on χ, we get

(2.61) $$\|F_3\|_{0,\tilde{\Omega}_{d/2}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

Let us now study F_4. From (2.56), (2.45), and the decomposition

$\chi_2 = \psi + z_2$, we see that F_4 can be written as

(2.62) $$F_4 = \chi\left(\frac{\partial^2\psi}{\partial\tilde{x}^2} + \tilde{a}^2(\tilde{x})\frac{\partial^2\psi}{\partial\tilde{y}^2}\right) - \tilde{a}^2(\tilde{x})\left(2\chi'(\tilde{y})\frac{\partial z_2}{\partial\tilde{y}} + \chi''(\tilde{y})z_2\right).$$

Because $z_2 = 0$ on $\partial\tilde{\Omega}$, in particular in $\tilde{\Gamma}_1$, the Poincaré inequality asserts that

(2.63) $$\int_0^A\int_0^d |z_2(\tilde{x},\tilde{y})|^2 d\tilde{x}d\tilde{y} \le \frac{d^2}{2} |z_2|^2_{1,\tilde{\Omega}}.$$

As a consequence of (2.53) and (2.59) we have

(2.64) $$\|z_2\|_{1,\tilde{\Omega}} \le C\|\tilde{u}\|_{1,\tilde{\Omega}} .$$

Now from (2.62), (2.57), (2.58), (2.63), and (2.64) we obtain

(2.65) $$\|F_4\|_{0,\tilde{\Omega}_{d/2}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

Theorem 2.1 can now be applied to χ_3, the solution of problem (2.54), with $F_3 + F_4 \in L^2(\tilde{\Omega}_{d/2})$ bounded by (2.61) and (2.65), and we obtain

(2.66) $$\|\chi_2\|_{2,\tilde{\Omega}_d} \le \|\chi_3\|_{2,\tilde{\Omega}_{d/2}} \le Cd^{-1}\|\tilde{u}\|_{1,\tilde{\Omega}}.$$

Finally from $\tilde{u} = \chi_1 + \chi_2$, (2.52), and (2.66) we get (2.39), which implies (2.34). ∎

3. Special Methods for Problems with Straight Line Unidirectional Coefficients

In this section we propose and analyze methods based on special elements to solve the model problem

$$(3.1)\quad \begin{cases} Lu(x,y) \equiv -\frac{\partial}{\partial x}(a(x)\frac{\partial}{\partial x}u(x,y)) - \frac{\partial}{\partial y}(a(x)\frac{\partial}{\partial y}u(x,y)) = f(x,y) \quad \forall\ (x,y) \in \Omega \\ u(x,y) = 0 \quad \forall\ (x,y) \in \partial\Omega, \end{cases}$$

where $\Omega = \Omega_0 = (0,1)\times(0,1)$, $f \in L^2(\Omega)$, and $a \in L^{\infty}(\Omega)$ is a function of x only and satisfies (1.2). This is problem (1.1) with $a(x,y)$ a straight line unidirectional coefficient. We will present three approximation methods, prove they have the optimal rate of convergence, and discuss their merits.

3.1. Approximation Method I.

For $0 < h \le 1$, let $\mathcal{E}_h$ be a triangulation of Ω by (closed) curvilinear triangles T of diameter $\le h$, where by a curvilinear triangle $T \subset \Omega$ we mean the pre-image of an ordinary triangle $\tilde{T} \subset \tilde{\Omega}$ under the mapping (2.7). Corresponding to $\mathcal{E}_h$ we have a triangulation $\tilde{\mathcal{E}}_h$ of $\tilde{\Omega}$ by usual triangles. We assume that $\{\tilde{\mathcal{E}}_h\}_{0<h\le 1}$ satisfies a minimal angle condition,

$$(3.2)\qquad h_{\tilde{T}}/\rho_{\tilde{T}} \le \sigma \quad \forall\ \tilde{T} \in \tilde{\mathcal{E}}_h, \quad \forall\ 0 < h \le 1,$$

where for any bounded set $S \subset \mathbb{R}^2$,

$$(3.3)\qquad h_S = \text{diameter of } S$$

and

$$(3.4)\qquad \rho_S = \text{diameter of the largest disk contained in } \bar{S}.$$

With $\mathcal{E}_h$ we associate the space of approximating (or shape) functions

$$(3.5)\qquad \begin{aligned} S_h = \{v \in L^2(\Omega) :\ & v|_T \in \operatorname{span}\{1, \int_0^x \frac{dt}{a(t)}, y\} \quad \forall\ T \in \mathcal{E}_h, \\ & v \text{ is continuous at the nodes of } \mathcal{E}_h, \\ & v = 0 \text{ at the boundary nodes}\}. \end{aligned}$$

As a consequence of our choice for the curvilinear triangles T we see that $S_h \subset H_0^1(\Omega)$, i.e., S_h is conforming. This is easily seen by noting that the functions $1, \int_0^x \frac{dt}{a(t)}, y$ are transformed to $1, \tilde{x}, \tilde{y}$ by (2.7). Consequently $\tilde{S}_h \equiv \{\tilde{v} : v \in S_h\}$ ($\tilde{v}$ is defined in (2.8)), the image of S_h under the mapping (2.7), is the usual space of continuous piecewise linear approximating functions with respect to $\tilde{\mathcal{E}}_h$, and S_h is conforming since $\tilde{S}_h$ is.

Our finite element approximation u_h to u is now defined by

$$(3.6) \qquad \begin{cases} u_h \in S_h \\ B(u_h, v) = \int_\Omega fv \, dxdy \quad \forall \ v \in S_h, \end{cases}$$

where B is defined in (1.3). u_h is just the Ritz approximation to u determined by the variational formulation (1.3), in the case (3.1), and the space S_h defined in (3.5). Since it is easily seen that $\tilde{u}_h$ is the Ritz approximation to $\tilde{u}$ determined by the variational formulation (2.9) and by the space $\tilde{S}_h$, we could, of course, carry out the computation and the analysis on the transformed domain $\tilde{\Omega}$. We shall however study the approximation on the original domain Ω since this approach better illuminates the more general case of a curvilinear unidirectional coefficient studied in Section 4.

It is immediate that B is a bounded bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. Furthermore, the stability condition (cf. [1]) holds, i.e., we have

<u>Theorem 3.1</u>. There exists a constant $\delta(\alpha) > 0$, independent of h, such that for all $0 < h \le 1$,

$$(3.7) \qquad \inf_{\substack{v \in S_h \\ \|v\|_{1,\Omega} = 1}} \ \sup_{\substack{w \in S_h \\ \|w\|_{1,\Omega} = 1}} |B(v,w)| \ge \delta(\alpha).$$

Proof. Since $B(v,w)$ is symmetric it is sufficient to prove that B is coercive, i.e., that

$$|B(v,v)| \geq \delta(\alpha)\ \|v\|^2_{1,\Omega}\ \forall v \in S_h,\ 0 < h \leq 1.$$

This is immediate.

Approximability here involves the approximation of the solution u by a linear combination of the shape functions $1, \int_0^x \frac{dt}{a(t)}, y$ in terms of which S_h is defined. Let the points $P_1, P_2, P_3 \in \Omega$ be the vertices of T and let $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ be the vertices of $\tilde{T}$ (cf. Fig. 3.1). Since the functions $1, \int_0^x \frac{dt}{a(t)}, y$ are transformed to $1, \tilde{x}, \tilde{y}$ by (2.7), we see that the interpolation problem: Given numbers w_1, w_2, w_3, find

$$w(x,y) = \alpha + \beta \int_0^x \frac{dt}{a(t)} + \gamma y \tag{3.8}$$

satisfying $w(P_i) = w_i$, $i = 1,2,3$, is uniquely solvable.

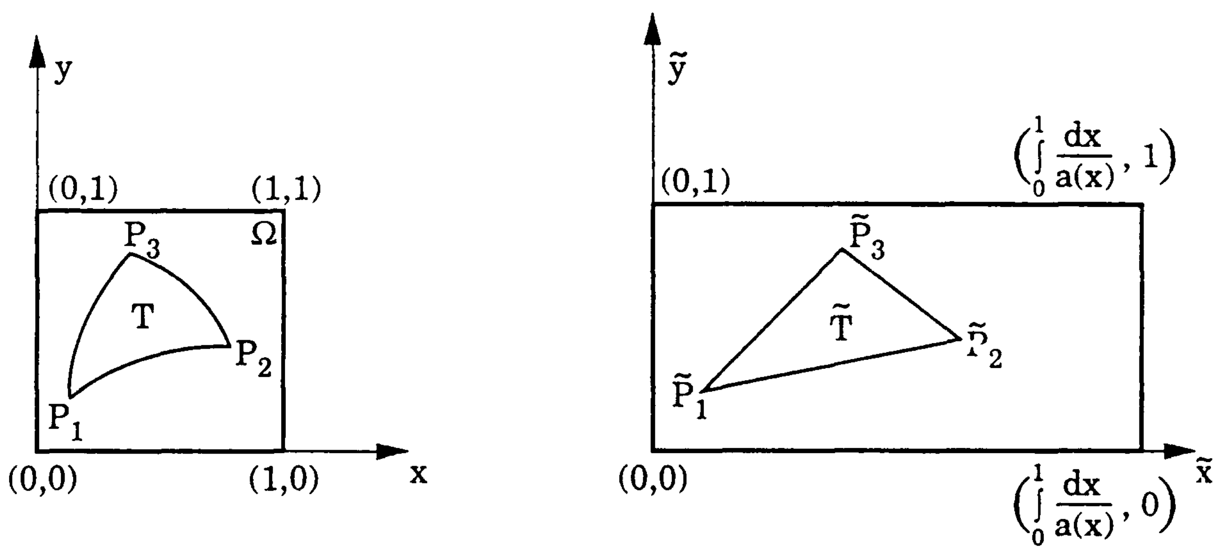

Fig. 3.1

Suppose $u \in H^L(T)$. Then $\tilde{u} \in H^2(\tilde{T})$, and hence $\tilde{u}$ has well defined point values for any $\tilde{P} \in \tilde{T}$. Thus u has well defined point values for any $P \in T$,

and we define the span$\{1, \int_0^x \frac{dt}{a(t)}, y\}$-interpolant of u on T by

$$d_T u = \alpha + \beta \int_0^x \frac{dt}{a(t)} + \gamma y, \quad d_T u(P_i) = u(P_i).$$

We derive now an estimate for the difference $u - d_T u$.

Theorem 3.2. There is a constant $C = C(\alpha, \beta)$, depending on α, β but independent of T and u, such that

$$(3.9) \qquad |u - d_T u|_{1,T} \le C \frac{h_{\tilde{T}}^2}{\rho_{\tilde{T}}} |u|_{L,T} \quad \forall\, u \in H^L(T),$$

where $h_{\tilde{T}}, \rho_{\tilde{T}}$ are defined in (3.3), (3.4).

Proof. Using the transformation (2.7), we have

$$(3.10) \qquad |u - d_T u|_{1,T}^2 = \int_T |\underset{\sim}{\operatorname{grad}}(u - d_T u)|^2 dx dy = \int_{\tilde{T}} \frac{1}{\tilde{a}} \left|\frac{\partial}{\partial \tilde{x}} (\tilde{u} - d_{\tilde{T}} \tilde{u})\right|^2 d\tilde{x} d\tilde{y}$$

$$+ \int_{\tilde{T}} \tilde{a} \left|\frac{\partial}{\partial \tilde{y}} (\tilde{u} - d_{\tilde{T}} \tilde{u})\right|^2 d\tilde{x} d\tilde{y}$$

$$\le \max(\beta, \frac{1}{\alpha}) |\tilde{u} - d_{\tilde{T}} \tilde{u}|_{1,\tilde{T}}^2,$$

where $d_{\tilde{T}} \tilde{u}$ is the span$\{1, \tilde{x}, \tilde{y}\}$-interpolant of $\tilde{u}$ in the triangle $\tilde{T}$. Applying the usual linear interpolation theorem (cf. [7, p. 121]), we get the bound

$$(3.11) \qquad |\tilde{u} - d_{\tilde{T}} \tilde{u}|_{1,\tilde{T}} \le C \frac{h_{\tilde{T}}^2}{\rho_{\tilde{T}}} |\tilde{u}|_{2,\tilde{T}},$$

where C is an absolute constant. Inequality (3.9) is a consequence of (3.10), (3.11), and the definition of the semi-norm $|\cdot|_{L,T}$, with the constant $C(\max(\beta, \frac{1}{\alpha}))^{1/2}$. ∎

We define now the S_h-interpolant of $u \in H^L(\Omega)$ by

$$\begin{cases} d_h u \in S_h \\ d_h u(P) = u(P) \quad \text{for all nodes } P \in \mathscr{E}_h. \end{cases} \tag{3.12}$$

As an easy corollary of Theorem 3.2, we can state our approximability result.

Theorem 3.3. There is a constant $C = C(\alpha,\beta,\sigma)$, depending on α,β, and σ but independent of u and h, such that

$$\|u - d_h u\|_{1,\Omega} \le Ch|u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \quad 0 < h \le 1. \tag{3.13}$$

Proof. Since the function $u - d_h u$ is in $H_0^1(\Omega)$, from the Poincaré inequality we have

$$\|u - d_h u\|^2_{1,\Omega} \le C(\Omega) \sum_{T\in\mathscr{E}_h} |u - d_T u|^2_{1,T}. \tag{3.14}$$

Combining (3.9), (3.12), and (3.14) we get

$$\|u - d_h u\|^2_{1,\Omega} \le C \sum_{T\in\mathscr{E}_h} \frac{h^4_{\tilde{T}}}{\rho^2_{\tilde{T}}} |u|^2_{L,T} \le C\,\sigma^2 \max_{T\in\mathscr{E}_h} h^2_{\tilde{T}}\, |u|^2_{L,\Omega}. \tag{3.15}$$

It follows immediately from the definition of the mapping (2.7) that

$$h_{\tilde{T}} \le \max(\tfrac{1}{\alpha}, 1) h_T \le \max(\tfrac{1}{\alpha}, 1) h \quad \forall\, T \in \mathscr{E}_h. \tag{3.16}$$

Finally, estimate (3.13) follows directly from (3.15) and (3.16). ∎

As a consequence of the stability, approximability, and regularity results, we obtain an estimate for the error $u - u_h$ in the $H^1(\Omega)$-norm.

Theorem 3.4. For $f \in L^2(\Omega)$ let u be the solution to (3.1) and let u be the solution to (3.6), with S_h defined in (3.5). Then there is a constant

$C = C(\alpha,\beta,\sigma)$, depending on α,β, and σ but independent of f and h, such that

(3.17) $$\|u-u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega}, \quad 0 < h \le 1.$$

Proof. It follows from Theorem 3.1 and standard results on the approximation of problems in variational form that

(3.18) $$\|u-u_h\|_{1,\Omega} \le C \inf_{\chi\in S_h} \|u-\chi\|_{1,\Omega}.$$

Combining (2.6), (3.13), and (3.18), and the fact that $u \in H_0^1(\Omega)$ implies $d_h u \in S_h$, we have

$$\|u-u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega},$$

where $C = C(\alpha,\beta,\sigma)$. ∎

Theorem 3.4 shows that the method defined by (3.6) is accurate and robust for the approximation of (3.1), i.e., the convergence is of first order in the mesh parameter h with a constant depending on α and β, but otherwise independent of the coefficient $a(x)$. Thus the method has the same accuracy as the usual finite element method based on C^0, piecewise linear approximating functions for smooth problems.

Remark 3.1 Approximation Method I, as we have presented it, is based on a triangular mesh. One can also consider rectangular meshes. Thus for $0 < h \le 1$, let $\mathcal{E}_h$ be a partition of Ω by rectangles R of diameter $\le h$ and suppose $\{\mathcal{E}_h\}_{0<h\le 1}$ satisfies a "minimal angle condition" ((diam R/diam of largest disk contained in R) $\le \sigma\ \forall R \in \mathcal{E}_h$ and $\forall\ 0 < h \le 1$). With $\mathcal{E}_h$ we associate the approximating functions

$$S_h = \{v \in C^0(\bar{\Omega}):\ v|_R \in \text{span}\left\{1,\ \int_0^x \frac{dt}{a},\ y,\ y\int_0^x \frac{dt}{a}\right\}\ \forall R \in \mathcal{E}_h,$$

$$v = 0 \text{ on } \partial\Omega\} .$$

The finite element approximation u_h is defined by (3.6) with this choice for S_h. Then it is easily seen that the arguments used to prove Theorem 3.4 yield

$$\|u - u_h\|_{1,\Omega} \le C(\alpha,\beta,\sigma)h \; \|f\|_{0,\Omega}, \tag{3.17'}$$

the same estimate proved for triangular meshes.

Remark 3.2. Method I has an obvious one dimensional version. This one dimensional method differs from the standard finite element method based on C^0, piecewise linear approximating functions in that the coefficient $a(x)$ enters the finite element calculations via its element-by-element harmonic averages instead of via its averages. It is referred to as a generalized displacement method (cf. [4]). In the methods presented in this paper, the coefficient $a(x,y)$ enters the calculations via various element-by-element harmonic averages and averages, i.e., via various element-by-element moments of $1/a(x,y)$ and $a(x,y)$.

3.2. Approximation Method II.

In Method I we chose shape functions that closely approximated the unknown solution. We then used the same functions for test functions, and the stability condition was immediate. In order to ensure our methods were conforming, we used curvilinear triangles. In this subsection, we discuss a second method, employing the triangulation by ordinary triangles shown in Fig. 3.2, the trial functions used in Method I, and C^0 piecewise linear test functions. Now the trial space will be non-conforming, but the test space will be conforming.

For $h = \frac{1}{n}$, $n = 2,3,\ldots$, let $\mathcal{C}_h$ be the uniform triangulation of Ω, with nodes $(x_i,y_j) = (ih,jh)$, $i,j = 0,\ldots,n$, shown in Fig. 3.2.

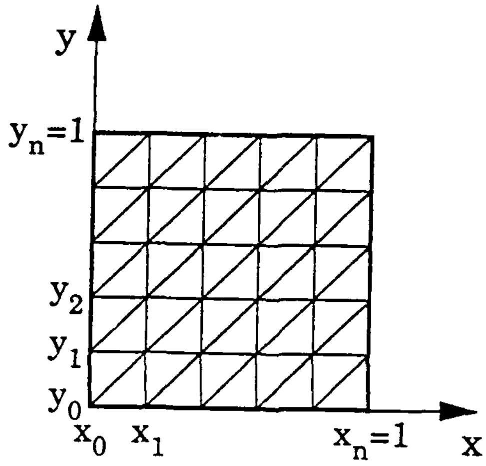

Fig. 3.2

For use in our analysis, we introduce the mesh dependent spaces

$$H_h^1(\Omega) = \{u \in L^2(\Omega) : u|_T \in H^1(T) \quad \forall\ T \in \mathscr{C}_h\} \tag{3.19}$$

with the norms

$$\begin{aligned} \|u\|_{1,h}^2 &= \int_\Omega u^2\, dxdy + |u|_{1,h}^2 \\ &= \int_\Omega u^2\, dxdy + \sum_{T\in\mathscr{C}_h} \int_T |\underset{\sim}{\mathrm{grad}}\ u|^2\, dxdy. \end{aligned} \tag{3.20}$$

It is clear that these spaces are Hilbert spaces.

We define the bilinear form B_h on $H_h^1(\Omega)\times H_0^1(\Omega)$ by

$$B_h(u,v) = \sum_{T\in\mathscr{C}_h} \int_T a\ \underset{\sim}{\mathrm{grad}}\ u \cdot \underset{\sim}{\mathrm{grad}}\ v\ dxdy. \tag{3.21}$$

Clearly B_h is bounded on $H_h^1(\Omega)\times H_0^1(\Omega)$, with a bound that is independent of h. Moreover, $B_h(u,v) = B(u,v)\ \forall\ u,v \in H_0^1(\Omega)$. Now we define the trial space

$S_{1,h}$ and the test space $S_{2,h}$:

(3.22) $$S_{1,h} = \Big\{ v \in L^2(\Omega) \;:\; v|_T \in \text{span}\{1, \int_0^x \frac{dt}{a(t)}, y\} \quad \forall\, T \in \mathcal{E}_h,$$
$$v \text{ is continuous at the nodes of } \mathcal{E}_h,$$
$$v = 0 \text{ at the boundary nodes} \Big\}$$

and

(3.23) $$S_{2,h} = \{ v \in C^0(\bar{\Omega}) \;:\; v|_T \in \text{span}\{1,x,y\},\; v|_{\partial\Omega} = 0 \}.$$

We remark that $S_{1,h} \not\subset H_0^1$ in general, so $S_{1,h}$ is non-conforming as mentioned above.

Our finite element approximation u_h to u is then defined by

(3.24) $$\begin{cases} u_h \in S_{1,h} \\ B_h(u_h, v) = \int_\Omega fv\,dx \quad \forall\, v \in S_{2,h}. \end{cases}$$

Note that the space $H_h^1(\Omega)$ is not well suited for a weak formulation of the exact problem (3.1). Nevertheless, the error analysis of (3.24) can be carried out in the usual way. Let us suppose that a stability condition holds for (3.24), i.e., there exists $\delta = \delta(\alpha,\beta)$ such that

(3.25) $$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{1,h}=1}} \; \sup_{\substack{v \in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(u,v)| \ge \delta(\alpha,\beta) > 0 \quad \forall\, 0 < h \le 1.$$

Since $\dim S_{1,h} = \dim S_{2,h}$, (3.25) implies that (3.24) is uniquely solvable. For any $u \in H_h^1(\Omega)$, we can define $P_h u$ by

(3.26) $$\begin{cases} P_h u \in S_{1,h} \\ B_h(P_h u, v) = B_h(u,v) \quad \forall\, v \in S_{2,h}. \end{cases}$$

It is clear that P_h is a projection onto $S_{1,h}$. This projection is uniformly bounded in h; in fact by (3.25) and (3.26) we have

$$\|P_h u\|_{1,h} \le \delta(\alpha,\beta)^{-1} \sup_{\substack{v\in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(P_h u,v)| \le C(\alpha,\beta)\|u\|_{1,h} \quad \forall\, u \in H_h^1(\Omega).$$

For u the solution of (3.1) and u_h the solution of (3.24) we have for any $\chi \in S_{1,h}$,

$$\|u-u_h\|_{1,h} = \|u-P_h u\|_{1,h} = \|(u-\chi)-P_h(u-\chi)\|_{1,h} \le [1+C(\alpha,\beta)]\|u-\chi\|_{1,h}.$$

Thus we have proved there exists a constant $C = C(\alpha,\beta)$ such that

(3.27) $$\|u-u_h\|_{1,h} \le C \inf_{\chi\in S_{1,h}} \|u-\chi\|_{1,h}$$

(cf. [1]).

We show now that the stability condition (3.25) holds.

Theorem 3.5. There is a positive constant $\delta = \delta(\alpha,\beta)$ such that

(3.28) $$\inf_{\substack{u\in S_{1,h} \\ \|u\|_{1,h}=1}} \sup_{\substack{v\in S_{2,h} \\ \|v\|_{1,\Omega}=1}} |B_h(u,v)| \ge \delta(\alpha,\beta) \quad \forall\, 0 < h \le 1.$$

Proof. Let $a_h : (0,1)\to\mathbb{R}$ denote the piecewise harmonic average of $a(x)$, i.e., let

(3.29) $$a_h|_{I_i} = \left\{h^{-1}\int_{x_{i-1}}^{x_i} \frac{dt}{a(t)}\right\}^{-1},$$

where $I_i = (x_{i-1},x_i)$. For any $u \in S_{1,h}$, let $v \in S_{2,h}$ be defined by

$$v(P) = u(P) \quad \text{for all nodes } P \text{ of } \mathscr{C}_h.$$

We will verify now the relations:

(3.30) $$a\frac{\partial u}{\partial x} = a_h\frac{\partial v}{\partial x},\quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}.$$

Let us first consider a triangle T of the type shown in Fig. 3.3.

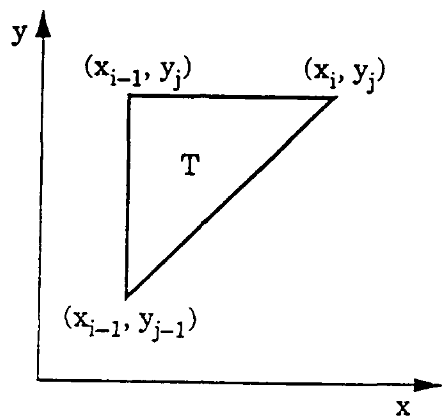

Fig. 3.3

On T we have

$$v(x,y) = u(x_{i-1},y_{j-1})\left\{1 - \frac{y-y_{j-1}}{h}\right\} + u(x_{i-1},y_j)\left\{\frac{(y-y_{j-1})-(x-x_{i-1})}{h}\right\}$$

$$+ u(x_i,y_j)\ \frac{x-x_{i-1}}{h}$$

and

$$u(x,y) = u(x_{i-1},y_{j-1})\left\{1 - \frac{y-y_{j-1}}{h}\right\} + u(x_{i-1},y_j)\left\{\frac{(y-y_{j-1})}{h} - \frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}\right\}$$

$$+ u(x_i,y_j)\ \frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}\ .$$

From these two formulae we clearly get

$$a\frac{\partial u}{\partial x} = \frac{u(x_i,y_j)-u(x_{i-1},y_j)}{h}\ \frac{h}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}} = a_h\ \frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} \quad \text{in} \quad T\ .$$

On the triangles of the type shown in Fig. 3.4 the proof follows the same

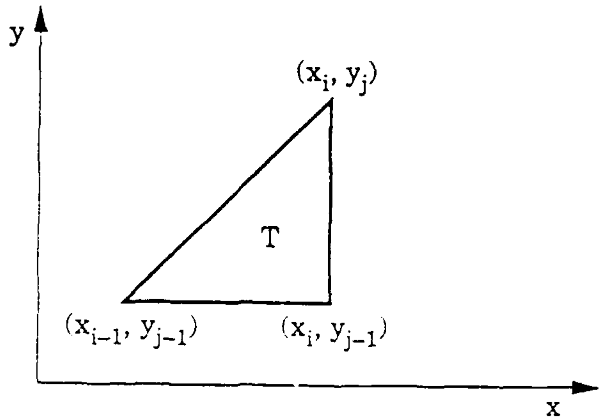

Fig. 3.4

lines. So the relations (3.30) are proved. Now using (3.30) and the Poincaré inequality we have

$$(3.31) \qquad B_h(u,v) = \sum_{T\in\mathscr{C}_h} \int_T \left\{a\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + a\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y}\right\}dxdy$$

$$\geq \alpha \sum_{T\in\mathscr{C}_h} \int_T |\underset{\sim}{\mathrm{grad}}\ v|^2\ dxdy = \alpha|v|^2_{1,\Omega} \geq \frac{\alpha}{2}\|v\|^2_{1,\Omega}.$$

To complete the proof we still have to bound $\|v\|_{1,\Omega}$ from below in terms of $\|u_1\|_{1,h}$. Using the relation (3.30) we obtain

$$(3.32) \qquad |u|^2_{1,h} = \sum_{T\in\mathscr{C}_h} \int_T \left\{\left(\frac{a_h}{a}\ \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}dxdy \leq \left(\frac{\beta}{\alpha}\right)^2 |v|^2_{1,\Omega}\ .$$

On the kind of triangles T shown in Fig. 3.3 we have

$$(3.33)\qquad \int_T |u|^2 dxdy = \int_T |u(x_{i-1},y_{j-1})\left\{1-\frac{y-y_{j-1}}{h}\right\}$$

$$+\,u(x_{i-1},y_j)\left\{\frac{y-y_{j-1}}{h}-\frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}\right\}+u(x_i,y_j)\frac{\int_{x_{i-1}}^{x}\frac{dt}{a(t)}}{\int_{x_{i-1}}^{x_i}\frac{dt}{a(t)}}|^2 dxdy$$

$$\le 3\int_T \{|u(x_{i-1},y_{j-1})|^2+|u(x_{i-1},y_j)|^2+|u(x_i,y_j)|^2\}dxdy$$

$$= 3\,\frac{h^2}{2}\,\{|u(x_{i-1},y_{j-1})|^2+|u(x_{i-1},y_j)|^2+|u(x_i,y_j)|^2\}$$

$$\le C\int_T |v|^2 dxdy\ .$$

On triangles of the type shown in Fig. 3.4 we have the same estimates. Inequalities (3.32) and (3.33) show that

$$(3.34)\qquad \|u\|_{1,h} \le \{C+\left(\frac{\beta}{\alpha}\right)^2\}^{1/2}\|v\|_{1,\Omega}.$$

From (3.31) and (3.34) it follows immediately that (3.28) holds with $\delta = \frac{\alpha}{2}\{C+\left(\frac{\beta}{\alpha}\right)^2\}^{-1/2}$. ∎

For $u \in H^L(\Omega)$, let $d_h u$ be the $S_{1,h}$-interpolant of u, i.e., let $d_h u$ be defined by

$$(3.35)\qquad \begin{cases} d_h u \in S_{1,h} \\ d_h u(P) = u(P) \quad \forall \text{ nodes } P \text{ of } \mathscr{C}_h; \end{cases}$$

$d_h u$ is well defined since u is well-defined on the nodes and since the images of the vertices of any $T \in \mathscr{C}_h$ are noncolinear. In the next theorem we derive an estimate for the interpolation error $\|u - d_h u\|_{1,h}$.

Theorem 3.6. There is a constant $C = C(\alpha,\beta)$, depending only on α and β but independent of u and h, such that

$$\|u - d_h u\|_{1,h} \le Ch|u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \quad 0 < h \le 1. \tag{3.36}$$

Proof. This proof is similar to that of Theorem 3.3. Let $u \in H^L(\Omega)$, $T \in \mathscr{C}_h$, and R_T be the smallest rectangle containing T. Let $\tilde{T},\tilde{R}_T$ be the images of T,R under the mapping (2.7). Then, applying the usual linear interpolation theorem as in (3.11), we have

$$|\tilde{u} - d_h\tilde{u}|_{i,\tilde{R}_T} \le C \frac{h^2_{\tilde{R}_T}}{\rho^i_{\tilde{R}_T}} |\tilde{u}|_{2,\tilde{R}_T}, \quad i = 0,1. \tag{3.37}$$

To obtain the result in the original variables we note that

$$|\tilde{u} - d_h\tilde{u}|^2_{0,\tilde{R}_T} = \int_{R_T} |u - d_h u|^2 \frac{1}{a}\, dxdy \ge \frac{1}{\beta}\, |u - d_h u|^2_{0,R_T} \tag{3.38}$$

and (cf. (3.10))

$$|\tilde{u} - d_h\tilde{u}|^2_{1,R_T} \ge \min(\alpha, 1/\beta)\ |u - d_h u|^2_{1,R_T}. \tag{3.39}$$

By the definition (2.5b) of the semi-norm $|\cdot|_{L,R_T}$, we have

$$|\tilde{u}|_{2,\tilde{R}_T} = |u|_{L,R_T}. \tag{3.40}$$

With (3.38) - (3.40), inequality (3.37) yields

$$|u - d_h u|_{i,T} \le C(\alpha,\beta) \frac{h^2_{\tilde{R}_T}}{\rho^i_{\tilde{R}_T}} |u|_{L,R_T}, \quad i = 0,1. \tag{3.41}$$

From (2.7) we have

$$h_{\tilde{R}_T} \le \max(1/\alpha, 1) h_{R_T}, \quad \rho_{\tilde{R}_T} \ge \min(1, 1/\beta)\ \rho_{R_T}. \tag{3.42}$$

Finally, (3.36) is a consequence of (3.41) and (3.42). ∎

As a consequence of (3.27), (3.28), (3.36), and (2.6) we obtain

<u>Theorem 3.7</u>. For $f \in L^2(\Omega)$, let u be the solution to (3.1) and let u_h be the solution to (3.24). Then there is a constant $C = C(\alpha, \beta)$ such that

$$\|u - u_h\|_{1,h} \le Ch\|f\|_{0,\Omega}. \tag{3.43}$$

<u>Remark 3.3</u>. In Remark 3.1 we briefly outlined Method I for rectangular meshes. Here we give a rectangular mesh version of Method II. Let

$$S_{1,h} = \left\{ v \in C^0(\bar{\Omega}) \;:\; v|_R \in \operatorname{span}\left\{1, \int_0^x \frac{dt}{a(t)}, \; y, \; y\int_0^x \frac{dt}{a(t)}\right\} \forall\, R \in \mathscr{C}_h, \right.$$
$$\left. v = 0 \quad \text{on} \quad \partial\Omega \right\}$$

and

$$S_{2,h} = \left\{ v \in C^0(\bar{\Omega}) \;:\; v|_R \in \operatorname{span}\left\{1, \; x, \; y, \; xy\right\} \forall\, R \in \mathscr{C}_h, \; v = 0 \quad \text{on} \quad \partial\Omega \right\}.$$

Our finite element approximation u_h is now defined by (3.24) with this choice for $S_{1,h}$ and $S_{2,h}$. In this situation we need a hypothesis on $a(x)$ in order to ensure stability. Let

$$\phi_i(x) = \frac{\displaystyle\int_{x_{i-1}}^{x} \frac{dt}{a}}{\displaystyle\int_{x_{i-1}}^{x_i} \frac{dt}{a}} \quad \text{and} \quad \psi_i = \frac{x - x_{i-1}}{h},$$

and then let

$$A_i = h^{-1} \int_{x_{i-1}}^{x_i} a \, dx,$$

$$B_i = h^{-1} \int_{x_{i-1}}^{x_i} \phi_i \, a \, dx,$$

$$C_i = h^{-1}\int_{x_{i-1}}^{x_i} \psi_i\,\phi_i\,a\,dx,$$

and

$$D_i = h^{-1}\int_{x_{i-1}}^{x_i} \psi_i\,a\,dx.$$

We assume

$$\frac{4A_iC_i - (B_i + D_i)^2}{4C_i} \ge \gamma > 0, \ \forall i,h.$$

Then (3.28) holds with $\delta = \delta(\alpha,\beta,\gamma) > 0$. We therefore obtain

$$\|u - u_h\|_{1,\Omega} \le C(\alpha,\beta,\gamma)h\ \|f\|_{0,\Omega}. \tag{3.43'}$$

We remark that $S_{1,h}$ is conforming in this rectangular mesh case in contrast with the triangular mesh case in which $S_{1,h}$ is nonconforming.

3.3. Approximation Method III.

In Method I we introduced curvilinear triangles in order to ensure the approximating functions were conforming, while in Method II we used a special triangulation with ordinary triangles obtaining a nonconforming method. In this section we design a conforming method based on an arbitrary triangulation with ordinary triangles.

For $0 < h \le 1$, let $\mathscr{C}_h$ be a triangulation of Ω by ordinary triangles of diameter $\le h$ and suppose $\{\mathscr{C}_h\}_{0<h\le1}$ satisfies

$$\frac{h_T}{\rho_T} \le \sigma \quad \forall \quad T \in \mathscr{C}_h, \ \forall\ h \quad \text{(minimal angle condition)} \tag{3.44}$$

and

$$\frac{h}{h_T} \le \nu \quad \forall \quad T \in \mathscr{C}_h, \ \forall\ h \quad \text{(quasi-uniform condition)}, \tag{3.45}$$

where h_T, ρ_T have been defined in (3.3), (3.4). Let $P_1, \ldots, P_{m_h}$ be the

nodes of $\mathscr{E}_h$. The function ψ_j denotes the usual piecewise linear basis function associated with the node $P_j = (x_j, y_j)$, $j = 1, \dots, m_h$, i.e.,

$$\psi_j \text{ is piecewise linear with respect to } \mathscr{E}_h,$$

$$\psi_j(P_i) = \delta_{ij};$$

we remark that

$$\sum_{j=1}^{m_h} \psi_j = 1.$$

For each $j \in \{1, \dots, m_h\}$ let

$$V_j = \operatorname{span}\{\psi_j(x,y), \psi_j(x,y) \int_{x_j}^{x} \frac{dt}{a(t)}, \ \psi_j(x,y)(y - y_j)\}.$$

For the space of approximating functions we choose

$$S_h = \{v : \Omega \longrightarrow \mathbb{R} : v = \sum_{j=1}^{m_h} v_j, \ \ v_j \in V_j, \ \ v = 0 \ \text{ on } \ \partial\Omega\}. \tag{3.46}$$

Our finite element approximation u_h to u is now defined by

$$\begin{cases} u_h \in S_h \\ B(u_h, v) = \int_\Omega f v \, dxdy \quad \forall \ v \in S_h. \end{cases} \tag{3.47}$$

u_h is the Ritz approximation to u determined by the variational formulation (1.3) and by the space S_h defined in (3.46). To study the convergence of the approximation (3.47), we turn our attention to an approximation result for $\{S_h\}_{0<h\le 1}$.

First we show that we can approximate $u \in H^L(\Omega)$ by a linear combination of $1, \int_{x_j}^{x} \frac{dt}{a(t)}, \ y - y_j$ on S_j, where for $j = 1, \dots, m_h$, S_j is the finite element star associated with the node P_j:

$$S_j = \bigcup_{\substack{T \in \mathcal{C}_h \\ P_j \in T}} T.$$

Let R_j be the smallest rectangle with sides parallel to the axes containing S_j and let J_j be three specific vertices of R_j, including all vertices that lie on $\partial\Omega$. For any $u \in H^L(R_j)$, we define the span$\{1, \int_{x_j}^{x} \frac{dt}{a(t)}, y - y_j\}$-interpolant of u associated with P_j by

$$\text{(3.48)} \qquad d_{j,h}u = \alpha + \beta \int_{x_j}^{x} \frac{dt}{a(t)} + \gamma(y - y_j), \quad (d_{j,h}u)(P) = u(P) \quad \forall\, P \in J_j.$$

We will prove the following approximability result.

<u>Theorem 3.8</u>. There is a constant $C = C(\alpha,\beta)$ such that

$$\text{(3.49)} \qquad |u - d_{j,h}u|_{i,S_j} \le C(\alpha,\beta)\, \frac{h^2_{R_j}}{\rho^i_{R_j}}\, |u|_{L,R_j}, \quad \forall i = 0,1, \quad j = 1,\dots,m_h, \quad u \in H^L(\Omega).$$

<u>Proof</u>. Let $j \in \{1,\dots,m_h\}$ and $u \in H^L(\Omega)$ be given. With the node P_j we associate the finite element star S_j and the rectangle R_j. $\tilde{S}_j$ and $\tilde{R}_j$ are the images under the mapping (2.7) of S_j and R_j. Clearly $\tilde{R}_j$ is also a rectangle. It follows from (2.7) and (3.48) that $\widetilde{d_{j,h}u}$ is the span$\{1,\tilde{x},\tilde{y}\}$-interpolant of $\tilde{u}$, i.e., $\widetilde{d_{j,h}u} \in \text{span}\{1,\tilde{x},\tilde{y}\}$ and $\widetilde{d_{j,h}u}(\tilde{P}) = \tilde{u}(\tilde{P})\ \forall\, P \in J_j$. Thus

$$\text{(3.50)} \qquad |\tilde{u} - \widetilde{d_{j,h}u}|_{i,\tilde{R}_j} \le C\, \frac{h^2_{\tilde{R}_j}}{\rho^i_{\tilde{R}_j}} |u|_{2,\tilde{R}_j}, \quad i = 0,1.$$

Returning to the original variables in (3.50) (cf. (3.38) - (3.40)), we obtain

$$(3.51) \qquad |u - d_{j,h}u|_{i,R_j} \le C(\alpha,\beta)\frac{h^2_{\tilde{R}_j}}{\rho^i_{\tilde{R}_j}}\,|u|_{L,R_j}, \quad i = 0,1.$$

As in (3.42) we have

$$(3.52) \qquad h_{\tilde{R}_j} \le \max(\frac{1}{\alpha},1)h_{R_j}, \quad \rho_{\tilde{R}_j} \ge \min(1,\frac{1}{\beta})\rho_{R_j}.$$

Finally, (3.49) is a consequence of (3.51) and (3.52). ∎

Before stating an approximation result for $\{S_h\}_{0<h\le1}$, we prove a technical result.

<u>Lemma 3.1</u>. Let $\{\mathscr{C}_h\}_{0<h\le1}$ be a family of triangulations satisfying the minimal angle condition (3.44). Let $P_1,\ldots,P_{m_h}$ denote the nodes of $\mathscr{C}_h$ and let S_j be the finite element star associated with P_j. Then we can partition the set $\{P_1,\ldots,P_{m_h}\}$ of nodes into a finite number of disjoint sets $I_1,\ldots,I_\ell$, with ℓ depending on σ but independent of h, such that $P_i,P_j \in I_k$, $i \ne j$, implies $\mathring{S}_i \cap \mathring{S}_j = \emptyset$ ($\mathring{S}_i$ denotes the interior of S_i).

<u>Proof</u>. The proof is simple; in fact, we give an algorithm to construct the partition. We assimilate the triangulation to a graph, the edges being arcs. Because of the minimal angle condition, a node P_i has a limited number of arcs $\overline{P_iP_{i_k}}$, $Q_i = \{P_{i_k} : k = 1,\ldots,\gamma_i\}$ being the neighbors of P_i, with $\gamma_i \le \gamma$, where γ depends on σ but is independent of i and h. We now state the algorithm. To construct I_1 we do the following. First take P_1 in I_1; then take the node of smallest index s in $\{P_1,\ldots,P_{m_h}\}\setminus(\{P_1\} \cup Q_1)$, to ensure $\mathring{S}_1 \cap \mathring{S}_s = \emptyset$, and so on until the set $\{P_1,\ldots,P_{m_h}\}\setminus(\{P_1\} \cup Q_1 \cup \{P_s\} \cup Q_s \cup \ldots)$ is empty. To construct I_2 we do the same as before with the subgraph of nodes $\{P_1,\ldots,P_{m_h}\}\setminus I_1$. From the minimal angle condition and the construction of I_1, a node of this subgraph will have at most $\gamma-1$

arcs. In this way we construct $I_3, \ldots$. The algorithm will stop after at most γ steps.∎

Theorem 3.9. There is a constant $C = C(\alpha,\beta,\nu,\sigma)$, depending on α,β,ν,σ but independent of u and h, such that

$$(3.53)\qquad |u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u|_{1,\Omega} \le Ch|u|_{L,\Omega} \quad \forall\, u \in H^L(\Omega), \quad 0 < h \le 1.$$

Proof. Let $u \in H^L(\Omega)$ and let $I_1,\ldots,I_\ell$ be the partition of the nodes of $\mathcal{C}_h$ given in Lemma 3.1. Then, since $\operatorname{supp} \psi_j = S_j$, we have

$$(3.54)\quad |u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u|^2_{1,\Omega} = |\sum_{j=1}^{m_h} \psi_j (u-d_{j,h}u)|^2_{1,\Omega}$$

$$= |\sum_{k=1}^{\ell} \sum_{j\in I_k} \psi_j (u-d_{j,h}u)|^2_{1,\Omega}$$

$$\le \ell \sum_{k=1}^{\ell} |\sum_{j\in I_k} \psi_j (u-d_{j,h}u)|^2_{1,\Omega}$$

$$= \ell \sum_{k=1}^{\ell} \int_\Omega |\sum_{j\in I_k} \underset{\sim}{\operatorname{grad}}(\psi_j (u-d_{j,h}u))|^2 \, dxdy$$

$$= \ell \sum_{k=1}^{\ell} \int_\Omega \sum_{j\in I_k} |(\underset{\sim}{\operatorname{grad}}\, \psi_j)(u-d_{j,h}u) + \psi_j\, \underset{\sim}{\operatorname{grad}}(u-d_{j,h}u)|^2 \, dxdy$$

$$\le 2\ell \sum_{j=1}^{m_h} \int_{S_j} \{|(\underset{\sim}{\operatorname{grad}}\, \psi_j)(u-d_{j,h}u)|^2 + |\psi_j\, \underset{\sim}{\operatorname{grad}}(u-d_{j,h}u)|^2\} dxdy.$$

We note that with the assumptions (3.44), (3.45), we have the bounds

$$|\psi_j| \le 1 \quad \text{and} \quad |\underset{\sim}{\text{grad}}\ \psi_j| \le \frac{1}{\min\limits_{P_j \in T} \rho_T} \le \frac{\sigma}{\min\limits_{P_j\in T} h_T} \le \frac{\sigma\nu}{h}.$$

Thus from (3.54) we get

$$(3.55) \qquad |u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u|^2_{1,\Omega} \le 2\ell \sum_{j=1}^{m_h} \left\{ \frac{1}{\min\limits_{P_j\in T} \rho_T^2} \int_{S_j} |u - d_{j,h}u|^2 dxdy + \int_{S_j} |\underset{\sim}{\text{grad}}(u - d_{j,h}u)|^2 dxdy \right\}.$$

We now use Theorem 3.8 in (3.55) to get

$$(3.56) \qquad |u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u|^2_{1,\Omega} \le 2\ell C(\alpha,\beta) \sum_{j=1}^{m_h} \left\{ \frac{h^4_{R_j}}{\min\limits_{P_j\in T} \rho_T^2} + \frac{h^4_{R_j}}{\rho^2_{R_j}} \right\} |u|^2_{L,R_j}.$$

With the assumptions (3.44) and (3.45), the following estimates are obvious:

$$h_{R_j} \le 2 \max_{P_j\in T} h_T \le 2h, \quad \rho_{R_j} \ge \min_{P_j\in T} \rho_T \ge \min_{P_j\in T} \frac{h_T}{\sigma} \ge \frac{h}{\nu\sigma}.$$

So the inequality (3.56) becomes

$$(3.57) \qquad |u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u|^2_{1,\Omega} \le C(\alpha,\beta,\sigma,\nu) h^2 \sum_{j=1}^{m_h} |u|^2_{L,R_j}.$$

It remains to estimate $\sum\limits_{j=1}^{m_h} |u|^2_{L,R_j}$. We have

$$(3.58) \qquad \sum_{j=1}^{m_h} |u|^2_{L,R_j} \le \sum_{T\in\mathscr{C}_h} N_T |u|^2_{L,T},$$

where

N_T = The number of rectangles R_j such that $T \cap R_j \ne \emptyset$.

Let us now show that under the assumptions (3.44) and (3.45) the numbers N_T can be bounded independently of h for all $T \in \mathscr{C}_h$. Let $T \in \mathscr{C}_h$ be given. If $T \cap R_j \neq \emptyset$, then P_j lies within the (closed) disk D of radius $(\sqrt{2}+1)h$ centered at the center of T. To estimate the number of nodes lying inside D, we first estimate the number $\bar{N}$ of triangles K that lie inside the disk D' of radius $(\sqrt{2}+2)h$. Since from (3.44), (3.45) we get

$$\frac{\pi h^2}{4\nu^2\sigma^2} \le \pi\frac{h_K^2}{4\sigma^2} \le \pi\,\frac{\rho_K^2}{4} \le \text{area}(K),$$

we have the estimate

$$\bar{N}\,\frac{\pi h^2}{4\nu^2\sigma^2} \le \sum_{K\subset D'} \text{area}(K) \le \text{area}(D') = \pi(2+\sqrt{2})^2h^2,$$

and hence

$$\bar{N} \le 4\nu^2\sigma^2(2+\sqrt{2})^2.$$

So for N_T we have the bound

$$N_T \le 12\nu^2\sigma^2(2+\sqrt{2})^2 \quad \forall\, T \in \mathscr{C}_h,\ 0 < h \le 1. \tag{3.59}$$

Finally combining (3.57)-(3.59) we get

$$\Big| u - \sum_{j=1}^{m_h} \psi_j d_{j,h} u \Big|_{1,\Omega}^2 \le Ch^2 |u|_{L,\Omega}^2, \tag{3.60}$$

where C depends on $\alpha, \beta, \nu, \sigma$ but not on u nor on h. Inequality (3.53) follows from (3.60). ∎

As with the Approximation Method I, the stability condition is immediate (cf. Theorem 3.1). In the same way we proved Theorem 3.4, we can prove

<u>Theorem 3.10</u>. For $f \in L^2(\Omega)$, let u be the solution to (3.1) and let u_h be the solution to (3.47). Then there is a constant $C = C(\alpha,\beta,\nu,\sigma)$ such

that

(3.61) $$\|u - u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega}.$$

Note that in the proof of Theorem 3.10 we use the fact that $u \in H_0^1(\Omega)$ implies $\sum_{j=1}^{m_h} \psi_j d_{j,h} u \in H_0^1(\Omega)$. This is true because J_j contains any vertices of R_j that lie on $\partial\Omega$.

3.4 Comments on Methods I, II, III

We have described three methods for approximating the solutions of problems of the type depicted in Fig. 1.1. The usual finite element method is inaccurate for these problems since the solutions may not be in $H^{1+\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Methods I and II are closely related. The central idea in these methods is to exploit the existence of a mapping from the general element to the reference element that transforms the special shape functions into polynomials and the unknown solution into a smooth function, and thereby obtain a good convergence rate. For singular corner behavior and homogeneous material, this idea is exploited in [5].

It is advantageous to use rectangular meshes in Ω that are aligned with the direction of the unidirectional composite, as described by $a(x,y)$, because they are the images of rectangular meshes on $\tilde{\Omega}$. The major difference between Methods I and II is in their treatment of the right hand side f. Since with Method II, f enters the computation through integrals of f times the usual piecewise linear test functions (as opposed to integrals of f times the special test functions (cf. (3.5)), Method II is preferable when many right hand sides must be treated. On the other hand, Method II is less stable than Method I, leading to larger constants in the error estimates (cf.

(3.17), (3.17′), (3.43), (3.43′)). In fact, for quadrilateral meshes Method II may not converge for some a's; see the hypothesis on $a(x)$ in Remark 3.3. We note that for triangular meshes Method II always converges.

Method III, although similar in its use of good local approximating functions, (e.g., functions satisfying the differential equation) has a rather different character than Methods I and II. In Method III the alignment of the mesh does not play a role. Finite element approximating spaces based on shape functions satisfying the differential equation have been suggested and employed in various contexts. The main problem in their use is the enforcement of some type of conformity. This can be done by various hybrid methods, e.g. (see [2], [12], e.g.). These are, however, problems in ensuring the stability of these methods, and some of these problems have not been satisfactorily resolved. In contrast, Method III proposed here has no problems of this type, and is very accurate and robust. For some computational aspects of a similar method employing harmonic polynomials in a p-version fashion and applied to the solution of Laplace's equation see [15].

4. Methods for Problems with Curvilinear Unidirectional Coefficients

The methods presented in Section 3 cover problems on rectangular domains with coefficients that globally vary sharply in at most one direction, i.e., that are straight line unidirectional. Here we extend Methods I and III to cover coefficients that locally vary sharply in at most one direction, i.e., that are curvilinear unidirectional, and to cover domains with curved boundaries. Method I′, the extension of Method I, will be based on quadrilateral and triangular elements and Method III′, the extension of Method III, will be based on triangular elements.

4.1. Method I′

Consider the boundary value problem (1.1) and suppose

- for $1 \le i \le n'$, $(\Omega_i, \xi_i, \eta_i)$ is an open subset of Ω and a coordinate system satisfying conditions (i)-(vi) in Section 2 and for $n' + 1 \le i \le n$, where $n' \le n$, $(\Omega_i, \xi_i, \eta_i)$ is an open subset of Ω and a coordinate system satisfying conditions (i)-(iii), (v), (vi) in Section 2, i.e., $\xi = \xi_i = \xi_i(x,y)$, $\eta = \eta_i = \eta_i(x,y)$ and if (x,y) ranges over Ω_i, then (ξ_i, η_i) ranges over $\Omega_i' = (\xi^1_{\Omega_i}, \xi^2_{\Omega_i}) \times (\eta^1_{\Omega_i}, \eta^2_{\Omega_i})$, where Ω_i, $\xi_i(x,y)$, $\eta_i(x,y)$ satisfy conditions (i)-(iv), (vi) if $i \le n'$ and conditions (i)-(iii), (vi) if $i \ge n' + 1$ (let E_i denote the union of the interior edges of Ω_i);
- $\{\Omega_i\}_{i=1}^n$ covers Ω in the sense that

$$\Omega = \cup_{i=1}^n \Omega_i \tag{4.1}$$

and

$$\partial\Omega = \cup_{i=1}^n \{\text{Interior } (\bar{\Omega}_i \cap \partial\Omega) \text{ in } \partial\Omega\}; \tag{4.2}$$

- for $1 \le i \le n'$, we have

$$a(x,y) = a(x_i(\xi,\eta), y_i(\xi,\eta)) = a_i(\xi) \ \forall\ (x,y) \in \Omega_i, \tag{4.3}$$

where $x = x_i(\xi,\eta)$, $y = y_i(\xi,\eta)$ is the inverse of the mapping $\xi_i(x,y)$, $\eta_i(x,y)$ (a_i here plays the role of a' in (2.12)), and for $n'+1 \le i \le n$, $a(x,y)$ is smooth on Ω_i.

With $\{(\Omega_i,\xi_i,\eta_i)\}_{i=1}^n$ satisfying these assumptions, for each i let $\mathcal{O}_i$ be the result of pulling each interior edge of Ω_i (cf. condition (vi) of Section 2) a distance d towards the center of Ω_i. Then the $\mathcal{O}_i$'s are open sets of the type considered in Theorem 2.3 (i.e., $\mathcal{O}_i \subset \Omega_i$, $\mathcal{O}_i \subset\subset \Omega_i$ if $\bar{\Omega}_i \cap \partial\Omega = \emptyset$ and $\partial\mathcal{O}_i \cap \partial\Omega_i \subset \partial\Omega$ if $\bar{\Omega}_i \cap \partial\Omega \ne \emptyset$) and $\{\mathcal{O}_i\}_{i=1}^n$ satisfies (4.1)-(4.2), provided d is sufficiently small. Note that $d = \text{dist}(\mathcal{O}_i, E_i)$. We consider d to be fixed.

We note that if (1.1) corresponds to problems of the type depicted in Figs. 1.1-1.3 or to a smooth interface problem modeled as a composite material, then $\{(\Omega_i,\ \xi_i,\ \eta_i)\}_{i=1}^n$ can be chosen to satisfy the conditions outlined above.

We note that these assumptions imply that $\partial\Omega$ is a piecewise smooth (C^2) curve with vertices with angular measure α satisfying $0 < \alpha < \pi$; in particular Ω has no reentrant vertices.

<u>Remark 4.1</u>. If our problem satisfies these assumptions we say that $a(x,y)$ locally varies sharply in at most one direction. Such coefficients are, as indicated earlier, also called (curvilinear) unidirectional.

With Ω, $\Omega_1,\dots,\Omega_n$, $\mathcal{O}_1,\dots,\mathcal{O}_n$, n', and $a(x,y)$ satisfying the hypothesis described above, we now describe the meshes we will employ. For $0 < h \le 1$, let $\mathcal{E}_h = \{T\}$ be a mesh on Ω consisting of curvilinear (closed) quadrilaterals or triangles, and satisfying the following properties:

- Each T is contained in some $\bar{\mathcal{O}}_i$: $T \subset \bar{\mathcal{O}}_{i(T)}$, $1 \le i(T) \le n$;
- If $i(T) \le n'$, then T is the image of a rectangle T' in $\Omega'_{i(T)}$

under the mapping $x = x_{i(T)} = x_{i(T)}(\xi,\eta)$, $y = y_{i(T)} = y_{i(T)}(\xi,\eta)$, i.e.,

$$T = \{(x,y) : x = x_{i(T)}(\xi,\eta),\ y = y_{i(T)}(\xi,\eta),$$

$$\xi^1_{\Omega_{i(T)}} \le \xi^1_T \le \xi \le \xi^2_T \le \xi^2_{\Omega_{i(T)}},\ \eta^1_{\Omega_{i(T)}} \le \eta^1_T \le \eta \le \eta^2_T \le \eta^2_{\Omega_{i(T)}}\},$$

where

$$|\xi^2_T - \xi^1_T| \le h, \quad |\eta^2_T - \eta^1_T| \le h, \tag{4.4a}$$

$$\sigma^{-1} \le \frac{|\xi^2_T - \xi^1_T|}{|\eta^2_T - \eta^1_T|} \le \sigma, \tag{4.4b}$$

where $1 \le \sigma < \infty$ is independent of the mesh. The mapping $(\xi_{i(T)}, \eta_{i(T)})$ maps T onto $T' = (\xi^1_T, \xi^2_T) \times (\eta^1_T, \eta^2_T)$ and T' is mapped onto the reference rectangle $T^0_r = (0,1) \times (0,1)$ by the mapping

$$\tilde{\xi} = \tilde{\xi}_{i(T)} = \frac{\displaystyle\int_{\xi^1_T}^{\xi} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi^1_T}^{\xi^2_T} \frac{dt}{a_{i(T)}}} = \frac{\displaystyle\int_{\xi^1_{\Omega_{i(T)}}}^{\xi} \frac{dt}{a_{i(T)}} - \int_{\xi^1_{\Omega_{i(T)}}}^{\xi^1_T} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi^1_T}^{\xi^2_T} \frac{dt}{a_{i(T)}}},\quad \tilde{\eta} = \tilde{\eta}_{i(T)} = \frac{\eta - \eta^1_T}{\eta^2_T - \eta^1_T}. \tag{4.5}$$

Thus the composition of these two mappings maps T onto $T^0 = T^0_r$, and the inverse, F_T, of the composition maps T^0 onto T.

- If $i(T) \ge n' + 1$, then T is the image of

$$T^0 = \begin{cases} T^0_r, & \text{if } T \text{ is a quadrilateral} \\ T^0_t = \text{a reference triangle}, & \text{if } T \text{ is a triangle} \end{cases}$$

under a mapping F_T satisfying

$$F_T \text{ is invertible, and } F_T \text{ and } F_T^{-1} \text{ are smooth,}$$

(4.6) $$|F_T|_{1,\infty,T^0} \le Ch, \quad |F_T|_{2,\infty,T^0} \le Ch^2, \quad |F_T^{-1}|_{1,\infty,T} \le Ch^{-1},$$

and

(4.7) $$|J_{F_T}|_{0,\infty,T^0} \equiv \sup_{(\tilde{\xi},\tilde{\eta})\in T^0} |J_{F_T}(\tilde{\xi},\tilde{\eta})| \le Ch^2, \quad |J_{F_T^{-1}}|_{0,\infty,T} \equiv \sup_{(x,y)\in T} |J_{F_T^{-1}}(x,y)| \le Ch^{-2},$$

where

$$|G|_{\ell,\infty,Q} = \sup_{(t,s)\in Q} \|D^{\ell}G(t,s)\|_{\mathcal{L}_{\ell}(R^2,R^2)},$$

$$\|D^{\ell}G(t,s)\|_{\mathcal{L}_{\ell}(R^2,R^2)} = \max_{\substack{\gamma_i\in R^2 \\ \|\gamma_i\|\le 1 \\ 1\le i\le \ell}} \|D^{\ell}G(t,s)\,(\gamma_1,\ldots,\gamma_{\ell})\|,$$

$$\|\cdot\| = \text{the Euclidean vector norm on } R^2,$$

and

$$J_G(t,s) = \text{Jacobian of } G \text{ at } (t,s).$$

The constant C in these estimates is independent of the mesh. We easily see that the mapping $F_T : T^0 \longrightarrow T$ defined above for $i(T) \le n'$ satisfies parallel assumptions. Hence we have $T = F_T(T^0)$ for all T, and it is convenient to associate the mesh $\mathcal{E} = \{T\}$ with the set of mappings $\{F_T\}$.

- The standard compatibility condition is satisfied. Suppose that T_1 and T_2 are quadrilaterals with a common edge ℓ : $\ell = \bar{T}_1 \cap \bar{T}_2$. See Fig. 4.1; note that we are using two copies of the reference rectangle T^0.

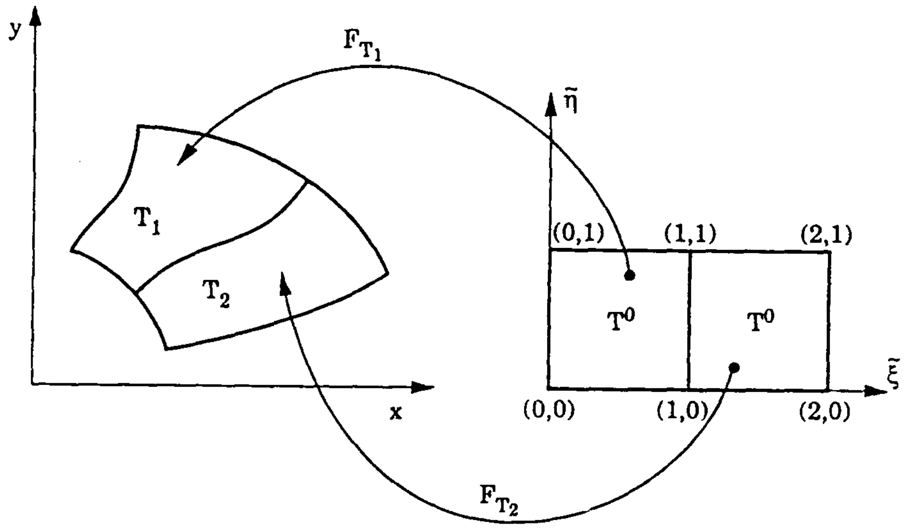

Fig. 4.1

Assuming that ℓ is the image of the vertical line segment $\{(\tilde{\xi},\tilde{\eta}) : \tilde{\xi} = 1, 0 \le \tilde{\eta} \le 1\}$ under both F_{T_1} and F_{T_2}, we require that

$$F_{T_1}(1,\tilde{\eta}) = F_{T_2}(1,\tilde{\eta}) \ , 0 \le \tilde{\eta} \le 1. \tag{4.8}$$

If ℓ is the image under F_{T_2} of a different edge of the reference rectangle, we would modify (4.8) in an obvious manner. Also, if either T_1 or T_2 is a triangle, the compatibility condition would be modified in an obvious way.

We point out that our mesh matches the (curved) boundary of Ω by means of blending (non-isoparametric) elements.

Remark 4.2. In the quadrilateral element case, verification of (4.6) and (4.7) usually proceeds along the following lines. Let T^* denote the straight line quadrilateral with vertices $a_i, i = 1,2,3,4$, coinciding with

those of T (see Fig. 4.2), let

$$h_T = \text{diam } T^*,$$

$$\rho_T = \text{diam of largest disk contained in } T^*,$$

$$\gamma_T = \max \{|\cos\{(a_{i+1} - a_i) \cdot (a_{i-1} - a_i)\}| \ 1 \le i \le 4 \pmod 4\}$$

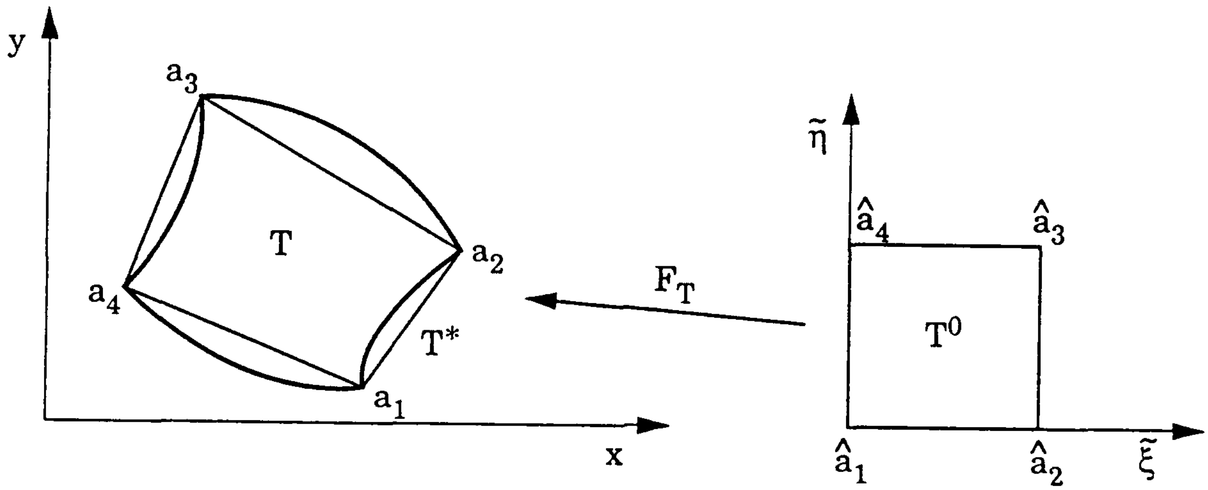

Fig. 4.2

and assume

$$h_T \le h, \ \frac{h_T}{\rho_T} \le \sigma, \ \gamma_T \le \gamma < 1,$$

where σ and γ are independent of the mesh. Let $\tilde{F}_T$ denote the bilinear mapping of T^0 onto T^* and write

$$F_T = \tilde{F}_T + \Phi.$$

One then makes assumptions on the perturbation Φ that imply (4.6) and (4.7) are satisfied. This procedure is outlined for isoparametric quadrilateral elements in [7, Exercise 4.3.9]. The parallel procedure for triangular isoparametric finite elements is carried out in [7, Theorem 4.3.3].

It remains to describe our shape functions. On $T \in \mathcal{E}_h$ we use the shape functions

$$(4.9) \qquad 1,\ F_{T,1}^{-1}(x,y),\ F_{T,2}^{-1}(x,y),\ F_{T,1}^{-1}(x,y)\, F_{T,2}^{-1}(x,y), \text{ if } T \text{ is a quadrilateral}$$

and

$$(4.9') \qquad 1,\ F_{T,1}^{-1}(x,y),\ F_{T,2}^{-1}(x,y), \text{ if } T \text{ is a triangle,}$$

where $F_T^{-1}(x,y) = (F_{T,1}^{-1}(x,y), F_{T,2}^{-1}(x,y))$, i.e., we use the pull-back polynomials determined by the bilinear shape functions $1, \tilde{\xi}, \tilde{\eta}, \tilde{\xi}\tilde{\eta}$ in the quadrilateral case and by the linear shape functions $1, \tilde{\xi}, \tilde{\eta}$ in the triangular case. For $i(T) \le n'$ we easily see that the functions in (4.9) are

$$(4.10) \quad 1,\ \frac{\displaystyle\int_{\xi_T^1}^{\xi_{i(T)}(x,y)} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}},\ \frac{\eta_{i(T)}(x,y)-\eta_T^1}{\eta_T^2 - \eta_T^1},\ \frac{\eta_{i(T)}(x,y)-\eta_T^1}{\eta_T^2 - \eta_T^1}\ \frac{\displaystyle\int_{\xi_T^1}^{\xi_{i(T)}(x,y)} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}}.$$

Then we let

$$(4.11) \qquad S_h = \{v \in L^2(\Omega) : v\big|_T \in \text{span of the shape functions on } T,$$
$$v \text{ is continuous at the nodes of } \mathcal{E}_h,$$
$$v = 0 \text{ at the boundary nodes}\}.$$

Because of the above assumptions, in particular (4.8), we see that $S_h \subset H_0^1(\Omega)$, i.e., S_h is conforming. The S_h-interpolant of u is defined by

$$(4.12) \qquad \begin{cases} d_h u \in S_h \\ (d_h u)(P) = u(P) \ \forall \text{ nodes } P \text{ of } \mathcal{E}_h. \end{cases}$$

Because of our choice of shape functions, $d_h u$ is a good approximation to u.

In Fig. 4.3 we show a typical part of the mesh on Ω. We show the sets Ω_i and O_i as well as the elements of the mesh. Note that both the Ω_i's and the elements fit the geometry of the fibers. In Fig. 4.4 we show the mesh in a neighborhood of the boundary of Ω. We see in particular the interior and the boundary edges of the Ω_i's. In Fig. 4.5 we show a typical mesh. We do not show the sets Ω_i and O_i, but do show the areas where the coefficient $a(x,y)$ is smooth and where it is rough (the areas with the fibers). Note that in the area of the fibers we use quadrilaterals elements while in the area where $a(x,y)$ is smooth we use both quadrilateral and triangular elements. Obviously triangular elements cannot be avoided, but quadrilateral elements are preferable because they usually lead to higher accuracy (although with the same rate of convergence).

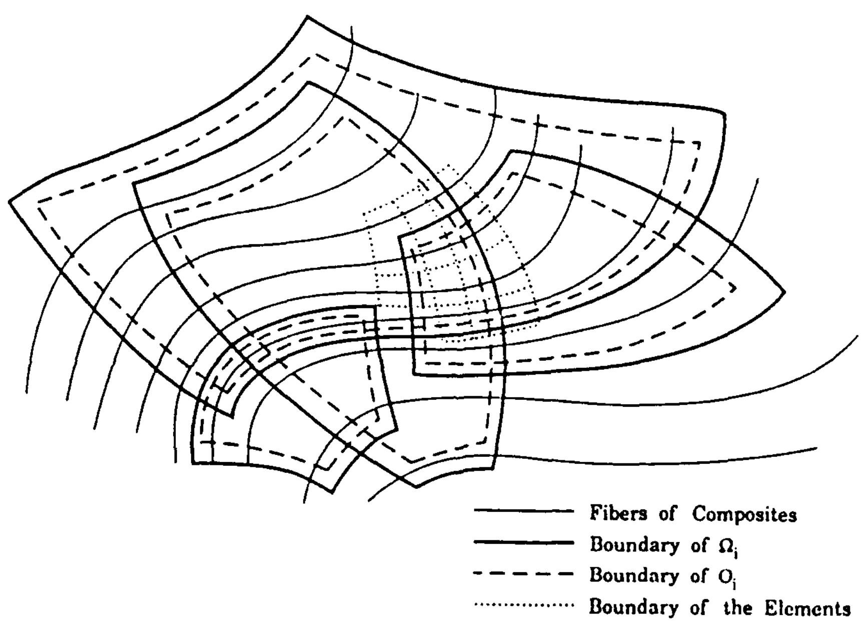

Fig. 4.3. Typical configuration of the sets Ω_i, O_i, and the elements inside Ω.

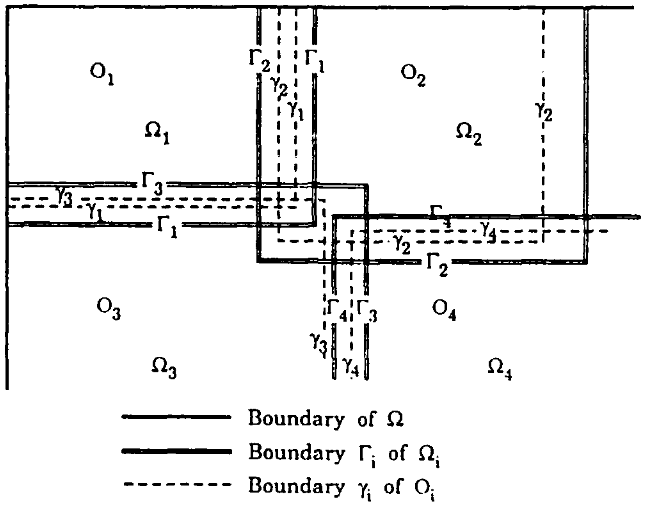

Fig. 4.4. Typical configuration of the sets Ω_i and O_i in the neighborhood of $\partial\Omega$.

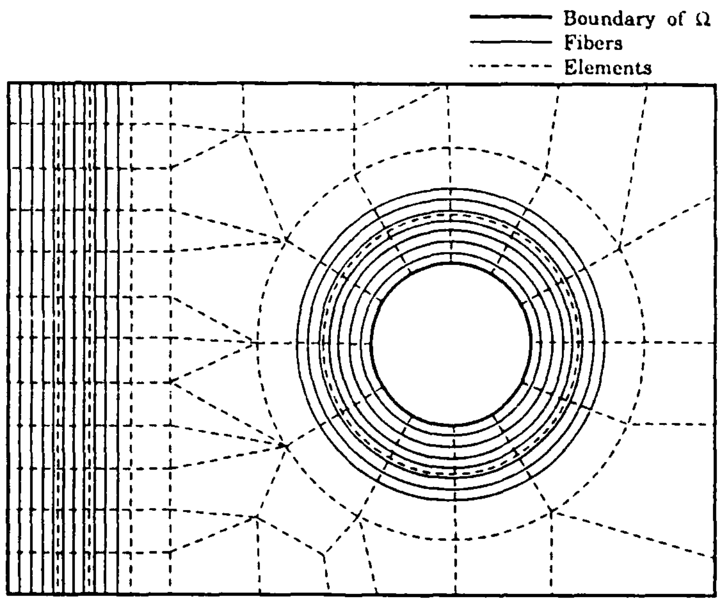

Fig. 4.5. A complete mesh for Method I'

The approximation property of the spaces S_h is formalized in

<u>Theorem 4.1</u>. There is a constant $C = C(\alpha, \beta, (\xi_1, \eta_1), \ldots, (\xi_{n'}, \eta_{n'}),$ $(\Omega_{n'+1}, \ldots, \Omega_n, a|_{\Omega_{n'+1}}, \ldots, a|_{\Omega_n}, d)$ depending on α, $\beta, (\xi_1, \eta_1), \ldots,$ $(\xi_{n'}, \eta_{n'})$, $\Omega_{n'+1}, \ldots, \Omega_n$, $a|_{\Omega_{n'+1}}, \ldots, a|_{\Omega_n}$, and d but independent of h and u, such that

$$\|u - d_h u\|_{1,\Omega} \le C\, h\, \|f\|_{0,\Omega}. \tag{4.13}$$

<u>Proof</u>. Consider $T \in \mathscr{C}_h$ and let $d_T u$ be defined by

$$d_T u \in \text{span}\,\{\text{shape functions on } T\},$$

$$(d_T u)(P) = u(P), \text{ for all vertices } P \text{ of } T$$

(cf(4.12)). For $i(T) \le n'$ we see that $d_T u$ is well defined by noting successively that $(d_T u)'$ (where the prime denotes the transformation from the variables (x,y) to the variables $(\xi_{i(T)}, \eta_{i(T)})$) is the

$$\text{span}\left\{ 1\ ,\ \frac{\displaystyle\int_{\xi_T^1}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}},\ \frac{\eta_{i(T)} - \eta_T^1}{\eta_T^2 - \eta_T^1},\ \frac{\eta_{i(T)} - \eta_T^1}{\eta_T^2 - \eta_T^1}\ \frac{\displaystyle\int_{\xi_T^1}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}}}{\displaystyle\int_{\xi_T^1}^{\xi_T^2} \frac{dt}{a_{i(T)}}} \right\}\text{-interpolant}$$

of u' at the points P', that $(d_{\tilde{T}} u)'$ (where the tilde denotes the transformation from the variables $(\xi_{i(T)}, \eta_{i(T)})$ to the variables

$$\tilde{\xi}_{i(T)} = \int_{\xi_{\Omega_i}^1}^{\xi_{i(T)}} \frac{dt}{a_{i(T)}},\ \tilde{\eta}_{i(T)} = \eta) \tag{4.14}$$

is the $\text{span}\{1, \tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}, \tilde{\xi}_{i(T)}\tilde{\eta}_{i(T)}\}$-interpolant of $\tilde{u}'$ at the points

$\tilde{P}'$, that $\tilde{u}' \in H^2(\tilde{\mathcal{O}}'_{i(T)})$, from Theorem 2.3, and that the points $\tilde{P}'$, for P a vertex of T, form the vertices of a rectangle in $\tilde{\mathcal{O}}'_{i(T)}$. Note that the variables $\tilde{\xi}_{i(T)}$ and $\tilde{\eta}_{i(T)}$ have here been defined differently than in (4.5). This is necessary in order that the set $\tilde{\mathcal{O}}'_{i(T)}$ depends only on $i(T)$ and not on T. For $i(T) \geq n' + 1$ we see that $d_T u$ is well-defined by noting successively that $\widetilde{d_T u}$ (where the tilde denotes the transformation from the variables (x,y) to the variables $\tilde{\xi} = F^{-1}_{T,1}(x,y)$, $\tilde{\eta} = F^{-1}_{T,2}(x,y))$ is the span$\{1, \tilde{\xi}, \tilde{\eta}, \tilde{\xi}\tilde{\eta}\}$-interpolant of $\tilde{u}$ if T is a quadrilateral and the span$\{1, \tilde{\xi}, \tilde{\eta}\}$-interpolant of $\tilde{u}$ if T is a triangle, that $\tilde{u} \in H^2(T^0)$, from standard elliptic regularity results since $T \subset \overline{\mathcal{O}_{i(T)}}$ and $a(x,y)$ is smooth on $\Omega_{i(T)}$, and that the points $\tilde{P}$ are the vertices of the rectangle T^0. We note that these observations show that $d_h u$ in (4.12) is well-defined.

Now we estimate $|u - d_T u|_{1,T}$. First suppose $i(T) \leq n'$. Changing variables we obtain

$$|u - d_T u|^2_{1,T} = \int_T \left\{ \left| \frac{\partial(u - d_T u)}{\partial x} \right|^2 + \left| \frac{\partial(u - d_T u)}{\partial y} \right|^2 \right\} dx\, dy$$

$$= \int_{T'} \left\{ \left| \frac{\partial(u' - (d_T u)')}{\partial \xi_{i(T)}} \right|^2 \left| \underset{\sim}{\text{grad}}\ \xi_{i(T)} \right|^2 \right.$$

$$\left. + \left| \frac{\partial(u' - (d_T u)')}{\partial \eta_{i(T)}} \right|^2 \left| \underset{\sim}{\text{grad}}\ \eta_{i(T)} \right|^2 \right\} \frac{\partial(x,y)}{\partial\left(\xi_{i(T)}, \eta_{i(T)}\right)}\, d\xi_{i(T)} d\eta_{i(T)}$$

$$\leq C \int_{T'} \left\{ \left| \frac{\partial(u' - (d_T u)')}{\partial \xi_{i(T)}} \right|^2 + \left| \frac{\partial(u' - (d_T u)')}{\partial \eta_{i(T)}} \right|^2 \right\} d\xi_{i(T)} d\eta_{i(T)}$$

$$= C\int_{\tilde{T}'}\left\{\frac{1}{\tilde{a}_{i(T)}}\left|\frac{\partial(\tilde{u}' - (d_T\tilde{u})')}{\partial\tilde{\xi}_{i(T)}}\right|^2 + \tilde{a}_{i(T)}\left|\frac{\partial(\tilde{u}' - (d_T\tilde{u})')}{\partial\tilde{\eta}_{i(T)}}\right|^2\right\} d\tilde{\xi}_{i(T)} d\tilde{\eta}_{i(T)}$$

$$\le C(\alpha,\beta,\xi_{i(T)},\eta_{i(T)})\ |\tilde{u}' - d_{\tilde{T}}\tilde{u}'|^2_{1,\tilde{T}'},$$

where $d_{\tilde{T}}\tilde{u}' = (d_T\tilde{u})'$ denotes the span$\{1, \tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}, \tilde{\xi}_{i(T)}\tilde{\eta}_{i(T)}\}$-interpolant of $\tilde{u}'$ on $\tilde{T}'$. Here $\tilde{\xi}_{i(T)}, \tilde{\eta}_{i(T)}$ are as defined in (4.14). Thus by standard approximation results for bilinear functions (cf. Theorem 3.1.4 in [7]) we have

$$|u - d_T u|_{1,T} \le C h\, |\tilde{u}'|_{2,\tilde{T}'}$$

and hence, for $1 \le j \le n'$,

$$\text{(4.15a)}\qquad \sum_{i(T)=j} |u - d_T u|^2_{1,T} \le C h^2 \sum_{i(T)=j} |\tilde{u}'|^2_{2,\tilde{T}'}$$

$$\le C h^2 |\tilde{u}'|^2_{2,\tilde{\mathcal{O}}_j}$$

$$= C h^2 |u|^2_{L,\mathcal{O}_j},$$

where $C = C(\alpha,\beta,(\xi_1,\eta_1),\dots,(\xi_{n'},\eta_{n'}))$. Now consider $i(T) \ge n' + 1$. Using (4.6) and (4.7) and the usual proof of approximation results (cf. proof of Theorem 4.3.4 in [7]), we obtain

$$|u - d_T u|_{1,T} \le C h(|u|_{1,T} + |u|_{2,T})$$

and hence, for $n' + 1 \le j \le n$,

$$\text{(4.15b)}\qquad \sum_{i(T)=j} |u - d_T u|^2_{1,T} \le C h^2 \sum_{i(T)=j} \|u\|^2_{2,T} \le C h^2 \|u\|^2_{2,\mathcal{O}_j}.$$

For any T, $(d_h u)\big|_T = d_T u$, and thus from (4.15a,b) we have

$$(4.16) \qquad |u - d_h u|^2_{1,\Omega} = \sum_{T \in \mathcal{E}_h} |u - d_T u|^2_{1,T}$$

$$= \sum_{1 \le j \le n'} \sum_{i(T)=j} |u - d_T u|^2_{1,T} + \sum_{n'+1 \le j \le n} \sum_{i(T)=j} |u - d_T u|^2_{1,T}$$

$$\le Ch^2 \left[\sum_{1 \le j \le n'} |u|^2_{L,\mathcal{O}_j} + \sum_{n'+1 \le j \le n} \|u\|^2_{2,\mathcal{O}_j} \right],$$

where $C = C(\alpha, \beta, (\xi_1, \eta_1), \ldots, (\xi_{n'}, \eta_{n'}))$. From Theorem 2.3 we have

$$(4.17) \qquad \sum_{1 \le j \le n'} |u|^2_{L,\mathcal{O}_j} \le C\|f\|^2_{0,\Omega} ,$$

where $C = C(\alpha, \beta, (\xi_1, \eta_1), \ldots, (\xi_{n'}, \xi_{n'}), d)$. Since $a(x,y)$ is smooth on Ω_j for $j \ge n' + 1$, from standard elliptic regularity results we have

$$(4.18) \qquad \sum_{n'+1 \le j \le n} \|u\|^2_{2,\mathcal{O}_j} \le C\|f\|^2_{0,\Omega} ,$$

where $C = C(\Omega_{n'+1}, \ldots, \Omega_n, a|_{\Omega_{n'+1}}, \ldots, a|_{\Omega_n}, d)$. As a direct consequence of (4.16)-(4.18) we get (4.13), as desired. ∎

Our finite element approximation u_h to u is now defined by

$$(4.19) \qquad \begin{cases} u_h \in S_h \\ B(u_h, v) = \int_\Omega fv dx \ \forall\ v \in S_h. \end{cases}$$

Since we are using S_h for both the test and trial space, stability is immediate. Approximability has been established in Theorem 4.1. We thus have

Theorem 4.2. Suppose $\Omega, (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)$, n', $a(x,y)$ satisfy the assumptions in the first part of this subsection. Suppose u is the solution of (1.1) and u_h is the solution of (4.19). Then there is a constant $C = C(\alpha, \beta, \sigma, (\xi_1, \eta_1), \ldots, (\xi_{n'}, \eta_{n'}), \Omega_{n'+1}, \ldots, \Omega_n, a|_{\Omega_{n'+1}}, \ldots, a|_{\Omega_n}, d)$ such that

$$\|u - u_h\|_{1,\Omega} \le Ch\ \|f\|_{0,\Omega}\ \forall\ f \in L^2(\Omega),\ 0 < h \le h_0. \tag{4.20}$$

4.2. Method III′

Consider the boundary value problem (1.1) and suppose $\{(\Omega_i, \xi_i, \eta_i)\}_{i=1}^{n'}$, $\{(\Omega_i, \xi_i, \eta_i)\}_{i=n'+1}^{n}$, $a(x,y)$, and $\{\mathcal{O}_i\}_{i=1}^{n}$ are as described in Subsection 4.1. For $0 < h \le h_0$ let $\mathcal{C}_h = \{T\}$ be a triangulation of Ω by ordinary triangles together with curvilinear triangles which fit the curved part of $\partial\Omega$, all of diameter $\le h$. For any $T \in \mathcal{C}_h$ let T^* be the ordinary triangle with the same vertices as T. Then $\mathcal{C}_h^* = \{T^*\}$ is a triangulation of Ω by ordinary triangles, but $\bigcup_{T^* \in \mathcal{C}_h^*} T^* = \bigcup_{T \in \mathcal{C}_h} T^*$ is a polygonal approximation to Ω and not an exact fit of Ω. We assume all T^* have diameter $\le h$ and that $\{\mathcal{C}_h^*\}_{0<h\le h_0}$ satisfies the minimal angle condition (3.44) and the quasi-uniform condition (3.45). Let $\{P_j = (x_j, y_j)\}_{j=1}^{m_h}$ be the nodes of $\mathcal{C}_h^*$ and let ψ_j denote the piecewise linear basis function corresponding to P_j (and the triangulation $\mathcal{C}_h^*$). If T is curvilinear, then by restricting the domain of definition of ψ_j or by extending ψ_j as a linear function we can assume ψ_j is linear on T, and hence that ψ_j is continuous on $\bar{\Omega}$ and linear on each T. Let $S_j = \bigcup_{\substack{T \in \mathcal{C}_h \\ P_j \in T}} T$ be the finite element star associated with P_j. Now it is easily seen that if h_0 is sufficiently small, then any S_j will lie in some $\bar{\mathcal{O}}_i$: $S_j \subset \bar{\mathcal{O}}_{i(j)}$, $1 \le i(j) \le n$. Let $S_j^{i(j)'} \subset \bar{\mathcal{O}}'_{i(j)}$ be the image of $S_j \cap \bar{\Omega}$ under the mapping $(\xi_{i(j)}, \eta_{i(j)})$, let $R_j^{i(j)'}$ be the smallest rectangle with sides parallel to the axes containing $S_j^{i(j)'}$, and let $R_j^{i(j)}$ be the preimage

of $R_j^{i(j)'}$ under $(\xi_{i(j)}, \eta_{i(j)})$. $R_j^{i(j)'} \subset \mathcal{O}'_{i(j)}$ since $\mathcal{O}'_{i(j)}$ is a rectangle, and hence $R_j^{i(j)}$ lies in $\mathcal{O}_{i(j)}$. Define $J_j^{i(j)}$ to be three specific vertices of $R_j^{i(j)}$, including all vertices that lie on $\partial\Omega$.

Next we define our space of approximating functions. For $j = 1,\dots,m_h$ let

(4.21)
$$V_j = V_j^{i(j)} = \begin{cases} \operatorname{span}\left\{\psi_j(x,y),\ \psi_j(x,y)\displaystyle\int_{\xi^1_{\Omega_i}}^{\xi_i(x,y)} \frac{dt}{a_i(t)},\ \psi_j(x,y)\ [\eta_i(x,y) - \eta^1_{\Omega_i}]\right\} \\ \qquad\qquad \text{if } 1 \le i = i(j) \le n' \\ \operatorname{span}\left\{\psi_j(x,y),\ \psi_j(x,y)\ (x-x_j),\ \psi_j(x,y)\ (y-y_j)\right\} \\ \qquad\qquad \text{if } n' + 1 \le i = i(j) \le n \text{ and } \bar{\Omega}_i \cap \partial\Omega = \emptyset \\ \operatorname{span}\left\{\psi_j(x,y),\ \psi_j(x,y)\ [\xi_i(x,y)-\xi^1_{\Omega_i}],\ \psi_j(x,y)\ [\eta_i(x,y)-\eta^1_{\Omega_i}]\right\} \\ \qquad\qquad \text{if } n' + 1 \le i = i(j) \le n \text{ and } \bar{\Omega}_i \cap \partial\Omega \ne \emptyset. \end{cases}$$

The 3rd line in this definition has been stated for the case in which the preimages of the points $(\xi^1_{\Omega_i}, \eta^1_{\Omega_i})$, $(\xi^2_{\Omega_i}, \eta^1_{\Omega_i})$ lie on $\partial\Omega$. In other situations we could modify the definition in an obvious way. Then for the space of approximating functions we choose

(4.22)
$$S_h = \{v : \Omega \to \mathbb{R} : v = \sum_{j=1}^{m_h} v_j,\ v_j \in V_j,\ \ v = 0 \text{ on } \partial\Omega\}.$$

Our finite element approximation u_h to u is now defined by

(4.23)
$$\begin{cases} u_h \in S_h \\ B(u_h, v) = \displaystyle\int_\Omega fv\ dxdy \quad \forall\ v \in S_h. \end{cases}$$

Since we are using the space S_h for both the test and the trial space, stability is immediate. To study the convergence of the method (4.23) we need to prove an approximation result for the spaces $\{S_h\}_{0<h\le 1}$. This is done by combining the ideas of Subsections 3.3 and 4.1.

Let $d_{j,h}^{i(j)}u$ be defined by

$$(4.24)\quad \begin{cases} d_{j,h}^{i(j)}u \in \begin{cases} \operatorname{span}\left\{1,\ \displaystyle\int_{\xi_{\Omega_i}^1}^{\xi_i(x,y)} \frac{dt}{a_i(t)},\ \eta_i(x,y) - \eta_{\Omega_i}^1\right\} & \text{if } 1 \le i = i(j) \le n' \\[2ex] \operatorname{span}\left\{1,\ x-x_j,\ y-y_j\right\} & \text{if } n'+1 \le i = i(j) \le n \\ & \text{and } \bar{\Omega}_i \cap \partial\Omega = \emptyset \\[2ex] \operatorname{span}\left\{1,\ \xi_i(x,y) - \xi_{\Omega_i}^1,\ \eta_i(x,y) - \eta_{\Omega_i}^1\right\} & \\ \quad \text{if } n'+1 \le i = i(j) \le n \text{ and } \bar{\Omega}_i \cap \partial\Omega = \emptyset. & \end{cases} \\[2ex] (d_{j,h}^{i(j)}u)\ (P) = u(P)\ \forall\ P \in J_j^{i(j)}. \end{cases}$$

$d_{j,h}^{i(j)}u$ is a good approximation to u on S_j, as made precise in

<u>Theorem 4.3</u>. There is a constant $C = C(\alpha, \beta, (\xi_1, \eta_1), \ldots, (\xi_n, \eta_n))$ such that

$$(4.25)\quad |u - d_{j,h}^{i(j)}u|_{k,S_j} \le \begin{cases} C\, \dfrac{h^2_{R_j^{i(j)}}}{\rho^k_{R_j^{i(j)}}}\, |u|_{L,R_j^{i(j)}}\ , & \text{if } 1 \le i(j) \le n' \\[3ex] C\, \dfrac{h^2_{R_j^{i(j)}}}{\rho^k_{R_j^{i(j)}}}\, \|u\|_{2,R_j^{i(j)}}\ , & \text{if } n'+1 \le i(j) \le n\ , \end{cases}$$

for $j = 1, \ldots, m_h$, $k = 0, 1$.

We omit the proof of this result since it is similar to that of Theorem 3.8.

The approximability result for $\{S_h\}_{0<h\le h_0}$ is given in

Theorem 4.4. There is a constant $C = C(\alpha,\beta,\nu,\sigma,(\xi_1,\eta_1),\dots,(\xi_n,\eta_n))$ such that

$$|u - \sum_{j=1}^{m_h} \psi_j d_{j,h}^{i(j)} u|_{1,\Omega} \le Ch\left[\sum_{i=1}^{n'} |u|_{L,\mathcal{O}_i} + \sum_{i=n'+1}^{n} \|u\|_{2,\mathcal{O}_i}\right]. \tag{4.26}$$

We omit the proof of this result since it is similar to that of Theorem 3.9.

Finally as a consequence of (4.26), Theorem 2.3, and standard elliptic regularity results we have

Theorem 4.5. Suppose Ω, $(\xi_1,\eta_1),\dots,(\xi_n,\eta_n)$, n', and $a(x,y)$ satisfy the assumptions in the first part of this subsection. Suppose u is the solution of (1.1) and u_h is the solution of (4.23). Then there is a constant $C = C(\alpha,\beta,\nu,\sigma,(\xi_1,\eta_1),\dots,(\xi_n,\eta_n),\ n',\ d)$ such that

$$\|u-u_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega} \ \forall\ f \in L^2(\Omega),\ 0 < h \le h_0. \tag{4.27}$$

4.3. Comments on Method I′ and III′

The differences and similarities of Methods I′ and III′ are similar to those of Methods I and III, which were discussed in Subsection 3.4. We note that with Method I′ we have to fit the elements to the geometry of the fibers of the composite, as seen in Fig. 4.5. This is not necessary in the case of Method III′, and this freedom could be utilized in many situations. For example, suppose the coefficient is changing rapidly but not abruptly along a line. Then Method III′ could be used, leading to an enrichment of the usual finite element space by special shape functions in the neighborhood of the line.

Implementational considerations and computational studies of Methods I, II, III, I′, and III′ will be presented in a forthcoming paper.

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Further information may be obtained from **Professor I. Babuška**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431.